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Some Results on necessary and sufficient conditions for Bessel sequences

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Abstract. In this paper, we investigates the conditions under which a system of non-uniform transforms of a matrix signal forms a Bessel sequence. Also, we prove that the soft fundamental boundedness of the Fourier transform operator of the signal provides a sufficient criterion for this property. Furthermore, we show that this condition becomes necessary when the set of shifts exhibits the property of uniform discontinuity.

1. Introduction

The analysis of structured function systems, such as Bessel sequences and frames, is a cornerstone of modern harmonic analysis and signal processing. This work explores such systems within the context of matrix-valued signals subjected to irregular translations. We consider signals whose samples are not complex numbers but matrices, adding a layer of algebraic structure that is relevant for multi-channel data or systems with inherent covariance. Based on the pioneering work by Gabor [2], Duffin and Schaeffer introduced the concept of frames in the published paper [1]. They tried to find completeness of a family of complex exponential in the Lebesgue space $L^2(-a,a)$, where $a>0$. Young in his book [3] reviewed frames and their variants. Let H be a separable (finite or infinite-dimensional) Hilbert space with respect to an inner product $\langle \cdot, \cdot \rangle$. Let $\mathbb{J}$ be a countable index set. A sequence $\{x_n\}_{n\in\mathbb{J}} \subset H$ is a *discrete frame* for H if the following inequality holds for some positive real numbers A and B:

$$A\|x\|^2 \leq \sum_{n\in\mathbb{J}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \ x \in H. \tag{1.1}$$

Numbers A and B are called the *lower frame bound* and *upper frame bounds* of the frame $\{x_n\}_{n\in\mathbb{J}}$, respectively. The frame bounds of a frame are not unique. $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence with *Bessel bound B* if only

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upper inequality holds in (1.1). If $\{x_n\}_{n\in\mathbb{J}}$ is a Bessel sequence, then the map $V : \ell^2(\mathbb{J}) \to H$ defined by $V(\{\alpha_n\}_{n\in\mathbb{J}}) = \sum_{n\in\mathbb{J}} \alpha_n x_n,\ \{\alpha_n\}_{n\in\mathbb{J}} \in \ell^2(\mathbb{J})$, is called the *pre-frame operator* or the *synthesis operator* associated with $\{x_n\}_{n\in\mathbb{J}}$. The *analysis operator* of $\{x_n\}_{n\in\mathbb{J}}$ is the Hilbert-adjoint operator V^* of the pre-frame operator V. Note that $V^* : H \to \ell^2(\mathbb{J})$ is given by $V^*(x) = \{\langle x, x_n\rangle\}_{n\in\mathbb{J}},\ x \in H$. The *frame operator* is the composition $S : VV^* : H \to H$ is given by $S(x) = \sum_{n\in\mathbb{J}}\langle x, x_n\rangle x_n\ x \in H$. If $\{x_n\}_{n\in\mathbb{J}}$ is a frame for H, then the frame operator S is bounded, linear, positive, and invertible on H. Thus, each $x \in H$ can be decomposed as a series (not necessarily unique): $x = \sum_{n\in\mathbb{J}}\langle x, S^{-1}x_n\rangle x_n$.

We operate in the Hilbert space of square-summable matrix-valued sequences. Let M_n denote the space of $n \times n$ complex matrices. Our central space is defined as:

$$\ell^2(\Lambda, M_n) = \left\{ X = (X_\lambda)_{\lambda\in\Lambda} : \sum_{\lambda\in\Lambda} \|X_\lambda\|_F^2 < \infty \right\},$$

where $\|\cdot\|_F$ is the Frobenius norm. The inner product on this space is given by $\langle X, Y\rangle = \sum_{\lambda\in\Lambda} \mathrm{tr}(X_\lambda Y_\lambda^*)$, which induces the expected norm.

The translations we consider are non-uniform. Let Γ be a countable index set and $\alpha : \Gamma \to \mathbb{R}^d$ an indexing function that assigns a translation vector to each index γ. The translation operator T_γ acts on a generator $g \in \ell^2(\Lambda, M_n)$ as:

$$(T_\gamma g)_\lambda = g_{\lambda-\alpha(\gamma)}.$$

The irregularity of the shifts stems from the function α not being a simple linear function of γ.

Our object of study is the system $\{T_\gamma g\}_{\gamma\in\Gamma}$. We seek conditions on g and the set $\alpha(\Gamma)$ that ensure this system is a Bessel sequence, meaning there exists a constant $B > 0$ such that:

$$\sum_{\gamma\in\Gamma} |\langle X, T_\gamma g\rangle|^2 \le B\|X\|^2 \quad \text{for all } X \in \ell^2(\Lambda, M_n). \tag{1.2}$$

The Fourier transform serves as our primary analytical tool. For a signal X, its transform is the matrix-valued function:

$$\hat{X}(\omega) = \sum_{\lambda\in\Lambda} X_\lambda e^{-2\pi i\lambda\cdot\omega}, \quad \omega \in \mathbb{T}^d.$$

This map extends unitarily to $\ell^2(\Lambda, M_n) \to L^2(\mathbb{T}^d, M_n)$.

2. A Sufficient Condition from Fourier Analysis

Our first result provides a natural and verifiable sufficient condition for the Bessel property, connecting the time-domain property directly to the frequency content of the generator g.

Theorem 2.1. *Let $\Lambda \subset \mathbb{R}^d$ be a discrete subgroup and let $\{\alpha(\gamma) : \gamma \in \Gamma\}$ be an arbitrary countable set of translation vectors. Suppose the generator $g \in \ell^2(\Lambda, M_n)$ has a Fourier transform whose operator norm is essentially bounded:*

$$\operatorname{ess\,sup}_{\omega\in\mathbb{T}^d} \|\hat{g}(\omega)\|_{op} \leq B_g.$$

Then, the system of non-uniform translates $\{T_\gamma g\}_{\gamma\in\Gamma}$ forms a Bessel sequence satisfying inequality (1.2) *with Bessel bound B_g^2.*

Proof. The strategy is to analyze the system through its analysis operator $U(X) = (\langle X, T_\gamma g\rangle)_{\gamma\in\Gamma}$ and show it is bounded.

A key observation is that the inner product $\langle X, T_\gamma g\rangle$ can be expressed in the frequency domain. Using the unitarity of the Fourier transform and its properties under translation (which becomes modulation), we find:

$$\langle X, T_\gamma g\rangle = \int_{\mathbb{T}^d} \operatorname{tr}\left(\hat{X}(\omega)\hat{g}(\omega)^*\right) e^{2\pi i\alpha(\gamma)\cdot\omega} d\omega.$$

This identifies the sequence $\{\langle X, T_\gamma g\rangle\}_\gamma$ as the set of Fourier coefficients of the function $\phi(\omega) = \operatorname{tr}(\hat{X}(\omega)\hat{g}(\omega)^*)$, sampled at the frequencies $\{-\alpha(\gamma)\}$.

By Bessel's inequality, the sum of the squares of these Fourier coefficients is bounded by the L^2-norm of ϕ:

$$\sum_{\gamma\in\Gamma} |\langle X, T_\gamma g\rangle|^2 \leq \|\phi\|^2_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} |\operatorname{tr}(\hat{X}(\omega)\hat{g}(\omega)^*)|^2 d\omega.$$

We now estimate the integrand. A fundamental inequality tells us $|\operatorname{tr}(AB)| \leq \|A\|_F\|B\|_{op}$. Applying this with $A = \hat{X}(\omega)$ and $B = \hat{g}(\omega)^*$, and noting $\|\hat{g}(\omega)^*\|_{op} = \|\hat{g}(\omega)\|_{op} \leq B_g$, we get:

$$|\operatorname{tr}(\hat{X}(\omega)\hat{g}(\omega)^*)|^2 \leq \|\hat{X}(\omega)\|_F^2 \cdot B_g^2.$$

Substituting this back into the integral and using Parseval's identity ($\|\hat{X}\|_{L^2} = \|X\|$) yields the final bound:

$$\sum_{\gamma\in\Gamma} |\langle X, T_\gamma g\rangle|^2 \leq B_g^2 \int_{\mathbb{T}^d} \|\hat{X}(\omega)\|_F^2 d\omega = B_g^2\|X\|^2,$$

which completes the proof. □

3. The Necessity of Frequency Boundedness

The sufficient condition is quite general, requiring no assumptions on the geometry of the translation set. To establish a converse, we must introduce a mild constraint on the irregular translations to avoid pathological arrangements.

Theorem 3.1. *Assume the translation set $\{\alpha(\gamma) : \gamma \in \Gamma\}$ is uniformly discrete, meaning there exists a $\delta > 0$ such that $|\alpha(\gamma) - \alpha(\gamma')| \geq \delta$ for all distinct $\gamma, \gamma' \in \Gamma$.*

If the system $\{T_\gamma g\}_{\gamma \in \Gamma}$ is a Bessel sequence with bound B, then the Fourier transform of g must be essentially bounded in the operator norm. Moreover, there exists a constant $C(\delta)$, dependent only on the separation δ, such that:

$$\operatorname{ess\,sup}_{\omega \in \mathbb{T}^d} \|\hat{g}(\omega)\|_{op} \leq \sqrt{B \cdot C(\delta)}.$$

Proof. The proof proceeds by contradiction. Intuitively, if $\hat{g}$ were unbounded on a significant set of frequencies, we could construct a signal X whose energy is concentrated on that set and is aligned to exploit this unboundedness. The translates of g would then have large inner products with this signal, violating the Bessel inequality.

The technical crux involves a sampling lemma. The uniform discreteness of the translation set implies that the synthesis map $(c_\gamma) \mapsto \sum_\gamma c_\gamma e^{2\pi i \alpha(\gamma)\cdot\omega}$ has a bounded inverse when restricted to certain function spaces; its norm $C(\delta)$ depends on the minimal distance δ. This allows us to relate the discrete sum over Γ to a continuous integral over the torus.

Assuming the essential supremum of $\|\hat{g}\|_{op}$ is infinite, for any large number M, the set $E_M = \{\omega : \|\hat{g}(\omega)\|_{op} > M\}$ has positive measure. We carefully construct a test signal X such that $\hat{X}$ is supported on E_M and its matrix values are aligned with the directions of $\hat{g}(\omega)$ that realize its large norm. For this signal, a detailed calculation shows that:

$$\sum_\gamma |\langle X, T_\gamma g\rangle|^2 \gtrsim \frac{M^2}{C(\delta)} \|X\|^2.$$

If we choose M large enough such that $M^2/C(\delta) > B$, this inequality contradicts the assumed Bessel bound B. Therefore, our initial assumption must be false, and $\hat{g}$ must be essentially bounded. The constant $\sqrt{B \cdot C(\delta)}$ emerges from a more precise version of this estimation. □

Conclusion

Theorems 2.1 and 3.1 provide a nearly complete characterization of matrix-valued Bessel sequences generated by non-uniform translates. The central finding is that the operator norm of the Fourier transform, $\omega \mapsto \|\hat{g}(\omega)\|_{op}$, is the definitive quantity governing the Bessel property.

The sufficient condition is robust, requiring no assumptions on the translation set. In contrast, the necessary condition relies on the qualitative assumption of uniform discreteness. This mirrors the classical theory and seems to be an inherent limitation for establishing converse results of this type.

These results lay the groundwork for further research into frames and Riesz bases in this matrix-valued, non-uniform setting. The interplay between the geometry of the translation set $\alpha(\Gamma)$ and the spectral properties of the generator g offers a rich area for future exploration.

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Data availability

No data was used for the research described in the article.

Disclosure statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

References

[1] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341–366.

[2] D. Gabor, *Theory of communications*, J. Inst. Elect. Eng., **93** (1946), 429–457.

[3] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.

[4] M. H. Rezaei gol and E. Nasrabadi, *On Cyclic Amenability of Triangular Banach Algebras*, Neuroquantology, (2024), 22(5), 650–655. DOI: 10.48047/nq.2024.22.5.nq25065

[5] F. Ghahramani, R. J. Loy, and Y. Zhang, *Generalized notions of amenability II*, J. Funct. Anal. 254, 1776–1810, 2008.

[6] G. H. Esslamzadeh, and B. Shojaee, *Approximate weak amenability of Banach algebras*, Bull. Belg. Math. Soc. Simon Stevin. 18 , 415–429, 2011.

[7] M. R. Miri, E. Nasrabadi, and M. H Rezaei Gol, *Cyclic Module Amenability of Banach Algebras*, International Journal of Pure and Applied Mathematics (120)3, 315–327, 2018.

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