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EQUITABLE ECCENTRIC DEGREE DOMINATION NUMBER OF GRAPHS

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Abstract: A subset $S \subseteq V(G)$ is called **an** *equitable eccentric degree dominating (EEDD) set if for every $v \in V - S$ there exists atleast one vertex $u \in S$ such that $uv \in E(G)$ and $|ed(u) - ed(v)| \leq 1$. Minimum cardinality of equitable eccentric degree dominating set is called equitable eccentric degree domination unmber (EEDDN) of G, denoted $\gamma^{ed}(G)$. In this paper we study (EEDDN) for path, complete graphs, cycles, star graphs, bipartite graphs and wheel graphs. We also obtain some bounds.*

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1-Introduction

Let $G = (V, E)$ be a simple connected graph having n vertices and m edges. For any vertex $v \in V(G)$, the degree of v, denoted by $d(v)$, is the number of edges incident with v. A graph G is called a regular graph if every vertex of G has the same degree. For two vertices $u, v \in V(G)$, the distance between u and v, written as $d(u, v)$, is the length of a shortest path joining them. The eccentricity of a vertex v, denoted by $e(v)$, is defined as $e(v) = max\, d(v, u) : u \in V(G)$. The radius of G, represented by $r(G)$, is the minimum eccentricity among all vertices of G, while the diameter of G, denoted by diam(G), is the maximum eccentricity among the vertices of G. A vertex v is said to be a central vertex if $e(v) = r(G)$. The collection of all central vertices forms the center of the graph and is denoted by $Z(G)$[4] .The double star graph $S_{m,n}$ is the graph constructed from $K_{1,m-1} and\ K_{1,n-1}$ by joining their centers with an edge [8]. A wheel graph $W_n, n \geq 5$ is a graph obtained by joining all vertices of cycle C_{n-1} to a further vertex called centre [10]. Domination in graph is one of the most active areas in graph theory, with applications ranging from network design and facility location to social influence and sensor coverage. The classical domination number $\gamma(G)$, defined as the minimum cardinality of a set $S \subseteq V(G)$ such that every vertex not in S has a neighbour in S, was formalised by [2,3,7]. E. Sampathkumar was the inaugural individual to acknowledge the essence and significance of this concept, subsequently introducing diverse forms of equitability in graphs, including degree equitability, outward equitability, inward equitability, equitability concerning the number of equal degree neighbors, and equitability

regarding the number of strong degree neighbors, among others [9]. The eccentric degree $ed(v)$ of a vertex v in graph G is the number of vertices at a distance equal to $e(v)$. The eccentric degree of G is the sum of eccentic degree of all vertices of G [6]. Among the many variants, Equitable domination [9] introduces a balance condition between the degree of a dominating vertex and the vertex it dominates, requiring $|deg(u) - deg(v)| \leq 1$.

Definition 1. [1] *A Prism graph $Y_{n,m}$ is a simple graph given by the Cartesian product graph $Y_{m,n} = C_m \times P_n$ where C_m is a cycle with m vertices and P_n is a path with n vertices. It can therefore be viewed and formed by connecting n concentric cycle graphs C_m along spokes. Also sometime $Y_{m,n}$ called a circular ladder.*

Definition 2. [9] *A subset $S \subseteq V(G)$ is called an equitable degree dominating set if for every $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such dominating set is denoted γ^e and is called equitable domination number of G.*

In general, if $G = (V, E)$ is a simple graph and $\phi: V(G) \to N$ is a function, we may define equitability of vertices in terms of ϕ - values of the vertices [9].

Theorem 1. [1] *For $m \geq 3$,*

$$\gamma(Y_{m,2}) = \begin{cases} 2\left\lfloor \frac{m-2}{4} \right\rfloor + 3, & \text{for } m \equiv 1(mod4) \\ 2\left\lfloor \frac{m-2}{4} \right\rfloor + 2, & \text{otherwise} \end{cases}$$

$$\gamma(Y_{m,3}) = \begin{cases} 3\left\lfloor \frac{m-2}{4} \right\rfloor + 4 & for\ m \equiv 1(mod4) \\ 3\left\lfloor \frac{m-2}{4} \right\rfloor + 2 & for\ m \equiv 2(mod4) \\ 3\left\lfloor \frac{m-2}{4} \right\rfloor + 3 & for\ m \equiv 0,3(mod4) \end{cases}$$

$$\gamma(Y_{m,4}) = \begin{cases} 4 & for\ m = 3 \\ 6 & for\ m = 5 \\ 10 & for\ m = 9 \\ m & for\ m \geq 4\ and\ m \neq 5,9. \end{cases}$$

$$\gamma(Y_{m,5}) = \begin{cases} 4 & for\ m = 3 \\ 8 & for\ m = 3 \\ \frac{5m}{4} & for\ m \geq 4, \quad m \neq 6\ and\ m \equiv 0(mod4) \\ \frac{5m+3}{4} & for\ m \geq 4, \quad m \neq 6\ and\ m \equiv 1(mod4) \\ \frac{5m-2}{4} & for\ m \geq 4, \quad m \neq 6\ and\ m \equiv 2(mod4) \\ \frac{5m+1}{4} & for\ \ m \geq 4, \quad m \neq 6\ \ and\ \ m \equiv 3(mod4) \end{cases}$$

2-EQUITABLE ECCENTRIC DEGREE DOMINATION NUMBER

Definition 3. *A subset $S \subseteq V(G)$ is called an equitable eccentric degree dominating (EEDD set) if for every $v \in V - S$ there exists atleast one vertex $u \in S$ such that $uv \in E(G)$ and $|ed(u) - ed(v)| \leq 1$. Minimum cardinality of equitable eccentric degree dominating set is called equitable eccentric degree domination number (EEDDN) of G and it is denoted by $\gamma^{ed}(G)$.*

Every EEDD set is a domination set and so, $\gamma(G) \leq \gamma^{ed}(G)$ but converse is not true. For example, consider the graph G given is Figure 1 ($ed(u) = 2$, $ed(v) = 3$, $ed(w) = ed(x) = 1$).

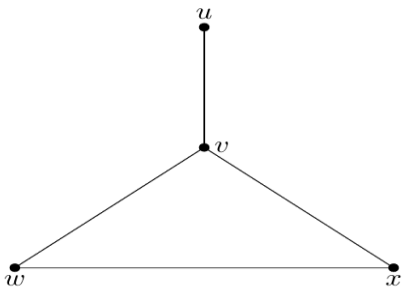


FIGURE 1. Graph G with vertices w, x, v, u

{v} is a dominating set but it is not an EEDD set because, w is adjacent to v but $|ed(v) - ed(w)| = 2 > 1$.

Proposition 2. *For the complete graph K_n, $\gamma^{ed}(K_n) = 1$.*

Proof. In K_n, $ed(v) = n - 1$, for vertex $v \in V(K_n)$. So, for every $u, v \in V(K_n)$, $|ed(u) - ed(v)| = 0$ and every vertex of K_n forms an EEDD set, Hence $\gamma^{ed}(K_n) = 1$.

Proposition 3. *For the path P_n, $\gamma^{ed}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof. The eccentric degree of any vertex of P_n is either 1 or 2 and $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Any dominating set of P_n is an EEDD set & hence

$\gamma^{ed}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proposition 4. *For the cycle C_n, $\gamma^{ed}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof. The eccentric degree of every vertex of $u \in V(C_n)$ is given by

$$ed(u) = \begin{cases} 2, & if\ n\ is\ odd \\ 1, & if\ n\ is\ even. \end{cases}$$

In either cases, $|ed(u) - ed(v)| = 0$, for any $u, v \in V(C_n)$. Also $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$

and hence $\gamma^{ed}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proposition 5. *For star graph $K_{1,n}$ $n \geq 2$, $\gamma^{ed}(K_{1,n}) = 1$.*

Proof. The central vertex u of $K_{1,n}$ has $ed(u) = n$ and the pendant vertex v_i, $1 \leq i \leq n$ has $ed(v_i) = n - 1$, So, $|ed(u) - ed(v_i)| = 1$ and $\{u\}$ is a dominating set of $K_{1,n}$. Hence $\gamma^{ed}(K_{1,n}) = 1$.

Proposition 6. *For wheel graph $W_n = C_{n-1} + K_1$, $n \geq 5$, $\gamma^{ed}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1$.*

Proof. In wheel graph $W_n = C_{n-1} + K_1$ $n \geq 5$, the central vertex u has $ed(u) = n - 1$ and the vertices v_i, $1 \leq i \leq n - 1$ on cycle C_{n-1} have $ed(v_i) = n - 3$. The set $\{u\}$ is a dominating set, but $|ed(u) - ed(v_i)| = 2$, So, it is not an EEDD set . Since $\gamma(C_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil$ it follows that these $\left\lceil \frac{n-1}{3} \right\rceil$ vertices together with u form minimum EEDD set, and so $\gamma^{ed}(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1$.

Proposition 7. *For complete bipartite graph $K_{m,n}$, $m, n \geq 2$*

$$ed(u) = \begin{cases} 2, & m - n \leq 1 \\ m + n & otherwise\ . \end{cases}$$

Proof. Let $V(K_{m,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots\ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots., v_n\}$, we have the following cases:
Case(1): Let $|m-n| \leq 1$, then any 2-element set $\{u_i, v_j\}$ forms an EEDD set and no singleten set forms an EEDD set. Hence $\gamma^{ed}(K_{m,n}) = 2$.
Case(2): Let $|m-n| \geq 2$. Then no proper subset of $V(K_{m,n})$ is an EEDD set & hence $\gamma^{ed}(K_{m,n}) = m+n$.

Proposition 8. *For prism graph $Y_{m,n} = C_m \times P_n$, $\gamma^{ed}(Y_{m,n}) = \gamma(Y_{m,n})$.*

Proof. The eccentric degree of every vertex of $u \in V(Y_{m,n})$ is given by

$$ed(u) = \begin{cases} 2, & if\ m\ is\ odd \\ 1, & if\ m\ is\ even. \end{cases}$$

For any two vertices $u, u \in V(Y_{m,n})$, $|ed(u) - ed(v)| = 0$, So remaining condition of equitable eccentric degree dominating set is same as that of the dominating set. So cardinality of minimum equitable eccentric degree dominating set of G is nothing but the domination number, and so, $\gamma^{ed}(Y_{m,n}) = \gamma(Y_{m,n})$

By theorem (1.1) and in [1]
For $m \geq 3$,

$$\gamma^{ed}(Y_{m,2}) = \begin{cases} 2\left\lfloor \frac{m-2}{4} \right\rfloor + 3 & for\ \ m \equiv 1(mod4) \\ 2\left\lfloor \frac{m-2}{4} \right\rfloor + 2 & otherwise. \end{cases}$$

$$\gamma^{ed}(Y_{m,3}) = \begin{cases} 3\left\lfloor \frac{m-2}{4} \right\rfloor + 4 & for\ m \equiv 1(mod4), \\ 3\left\lfloor \frac{m-2}{4} \right\rfloor + 2, & for\ m \equiv 2(mod4) \\ 3\left\lfloor \frac{m-2}{4} \right\rfloor + 3, & for\ m \equiv 0,3(mod4) \end{cases}$$

$$\gamma^{ed}(Y_{m,4}) = \begin{cases} 4 & for \quad m = 3 \\ 6 & for \quad m = 5 \\ 10 & for \quad m = 9 \\ m & for \quad m \geq 4 \ \ and \ \ m \neq 5,9. \end{cases}$$

$$\gamma^{ed}(Y_{m,5}) = \begin{cases} 4 & for\ m = 3 \\ 8 & for\ \ m = 6 \\ \frac{5m}{4} & for\ \ m \geq 4, m \neq 6\ and\ \ m \equiv 0(mod4) \\ \frac{5m+3}{4} & for\ m \geq 4, m \neq 6\ and\ \ m \equiv 1(mod4) \\ \frac{5m-2}{4} & for\ m \geq 4, m \neq 6\ and\ m \equiv 2(mod4) \\ \frac{5m+1}{4} & for\ m \geq 4, m \neq 6\ nand\ m \equiv 3(mod4) \end{cases}$$

Proposition 9. *Let $S_{m,n}$, $2 \leq m \leq n$ be a double star, $\gamma^{ed}(S_{m,n}) = 2$*

Proof. Let $V(S_{m,n}) = \{u, w, u_1, u_2, \dots, u_{m-1}, w_1, w_2, \dots, w_{n-1}\}$, where u and w are the central vertices, and $\{u_1, u_2, \dots, u_{m-1}\}$ and $\{w_1, w_2, \dots, w_{n-1}\}$ are the sets of pendant vertices adjacent to u and w, respectively. Then,
$ed(u) = n - 1, ed(w) = m - 1$, and $ed(u_i) = n - 1, 1 \leq i \leq m - 1$,
$ed(w_i) = m - 1, 1 \leq i \leq n - 1$. Then $\{u, w\}$ is a minimum dominating set such that $|ed(u) - ed(u_i)| = 0$ for every $u_i \in V(S_{m,n})$ and $|ed(w) - ed(w_i)| = 0$ for every $w_i \in V(S_{m,n})$ Hence $\{u, w\}$ is a minimum equitable eccentric degree dominating set & so, $\gamma^{ed}(S_{m,n}) = 2$

Theorem 10. *If $ed(u) = ed(v)$ for every edge $uv \in E(G)$, then $\gamma^{ed}(G) = \gamma(G)$.*

Proof. When all adjacent pair $\{u, v\}$ have equal eccentric degree then $|ed(u) - ed(v)| = 0$ for every edge u, v and every dominating set is EEDD set, so the two parameters coincide.

3-MINIMAL EQUITABLE ECCENTRIC DEGREE DOMINATING SETS

Definition 4. *A vertex $u \in V$ is called eccentric degree equitable with a vertex $v \in V$ if $|ed(u) - ed(v)| \leq 1$.*

Definition 5. *An equitable eccentric degree dominating set S is called a minimal equitable eccentric degree dominating set (or 1- minimal) if no proper subset of S is an equitable eccentric degree dominating set.*

Definition 6. *A minimal equitable eccentric degree dominating set of maximum cardinality is said to be a Γ^{ed}-set and its cardinality is denoted Γ^{ed}*

Theorem 11. *An equitable eccentric degree dominating set S is minimal if and only if for every vertex $u \in S$ one of the following holds.*

(1) Either $N(u) \cap S = \phi$ *or* $|ed(u) - ed(v)| \geq 2$ *for all* $v \in N(u) \cap S$.

(2) There exists a vertex $v \in V - S$ *such that* $N(v) \cap S = \{u\}$ *and* $|ed(u) - ed(v)| \leq 1$.

Proof. Suppose S is a minimal equitable eccentric degree dominating set. Assume (1) and (2) do not hold. Then for some $u \in S$ there exists $v \in N(u) \cap S$ such that $|ed(u) - ed(v)| \leq 1$ and for every $v \in V - S$, either $N(v) \cap S \neq \{u\}$ or $|ed(u) - ed(v)| \geq 2$ or both. Therefore $S - \{u\}$ is equitable eccentric degree dominating set, contradiction to the minimality of S. Therefore (1) or (2) holds.
Conversely, assume that for every $u \in S$, one of the statement (1) or (2) holds. Suppose S is not minimal. Then there exists $u \in S$ such that $S - \{u\}$ is an equitable eccentric degree dominating set. Therefore there exists a $v \in S - \{u\}$ such that v equitably eccentric degree dominates u. That is $v \in N(u)$
and $|ed(u) - ed(v)| \leq 1$. Therefore u does not satisfy (1). Then u must satisfy (2). Then there exists a $v \in V - S$ such that
$N(v) \cap S = \{u\}$ and $|ed(u) - ed(v)| \leq 1$. Since $S - \{u\}$ is an equitable eccentric degree dominating set, there exists $w \in S - \{u\}$ such that w is adjacent to v and w is equitable eccentric degree with v. Therefore $w \in N(v) \cap S$, $|ed(w) - ed(v)| \leq 1$ and $w \neq u$, a contradiction to $N(v) \cap S = \{u\}$. Therefore S is a minimal equitable eccentric degree dominating set.

4-EQUITABLE ECCENTRIC DEGREE INDEPENDENT SETS

Definition 7. *Let* $u \in V$. *The equitable eccentric degree neighbourhood of* u *denoted by* $N^{ed}(u)$ *is defined as* $N^{ed}(u) = \{v \in V / v \in N(u), |ed(u) - ed(v)| \leq 1\}$ *and* $u \in I_{ed} \Leftrightarrow N^{ed}(u) = \phi$, *where* I_{ed} *denoted the set of all equitable eccentric degree isolates.*

Definition 8. *The maximum and minimum equitable eccentric degree of a vertex in* G *are denoted respectively by* $\Delta^{ed}(G)$ *and* $\delta^{ed}(G)$.
That is $\Delta^{ed}(G) = max_{u \in V(G)} |N^{ed}(u)|$, $\delta^{ed}(G) = min_{u \in V(G)} |N^{ed}(u)|$.

Definition 9. *A subset* S *of* V *is called an equitable eccentric degree independent set, if for any* $u \in S$, $v \notin N^{ed}(u)$ *for all* $v \in S - \{u\}$.

Theorem 12. *Let* S *be a maximal equitable eccentric degree independent set. Then* S *is a minimal equitable eccentric degree dominating set.*

Proof. let S be a maximal equitable eccentric degree independent set. let $u \in V - S$. If $u \notin N^{ed}(v)$ for every $v \in S$, then $S \cup \{u\}$ is an equitable eccentric degree independent set, a contradiction to the maximality of S. Therefore $u \in N^{ed}(v)$ for some $v \in S$. Therefore S is an equitable eccentric degree dominating set. Since for any $u \in S$, $u \notin N^{ed}(v)$ it follows that for every $v \in S - \{u\}$, either $N(u) \cap S = \phi$ or $|ed(u) - ed(v)| \geq 2$ for all $v \in N(u) \cap S$. Therefore S is minimal equitable eccentric degree dominating set.

5-Bounds

Theorem 13. *Let G be a connected graph with n vertices and a maximum degree $\Delta(G)$. Then,* $\left\lceil \frac{n}{\Delta(G)+1} \right\rceil \leq \gamma^{ed}(G)$

Proof. Since every vertex of G can dominate at most $\Delta(G)$ vertices together with itself, any equitable eccentric degree dominating set must contain at least $\frac{n}{\Delta(G)+1}$ vertices. As $\gamma^{ed}(G)$ is an integer, we obtain $\gamma^{ed}(G) \geq \left\lceil \frac{n}{\Delta(G)+1} \right\rceil$. Hence, $\left\lceil \frac{n}{\Delta(G)+1} \right\rceil \leq \gamma_{ed}(G)$.

6-Conclusion

In this paper, we introduced and studied the equitable eccentric degree domination number for several classes of graphs. We obtained exact values and bounds for this parameter and established relationships with domination and equitable eccentric degree concepts. We also characterized minimal equitable eccentric degree dominating sets. These results provide a basis for further study of this domination parameter in other graph families.

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