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A Transform-Based Homotopy Analysis Method Using the Fractional Double Sumudu Transform for Nonlinear Fractional Partial Differential Equations

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Abstract

The study of fractional partial differential equations has received considerable attention because of their broad applicability in modeling complex systems exhibiting nonlocal and memory-dependent behaviour. In this paper, a new hybrid analytical technique, namely the Fractional Double Sumudu Homotopy Analysis Transform Method (FDSHATM) is proposed for obtaining approximate analytical solutions of nonlinear fractional partial differential equations. The method combines the Fractional Double Sumudu Transform with the Homotopy Analysis Method to generate rapidly convergent series solutions without discretization or perturbation assumptions. The proposed algorithm is applied to two FPDEs and the obtained results are found to be in excellent agreement with the corresponding exact solutions. The convergence of the solution series is effectively controlled through the auxiliary parameter $\hbar \neq 0$, demonstrating the accuracy, reliability and efficiency of the proposed method. The developed technique provides a simple and effective framework for solving a broad class of linear and nonlinear fractional partial differential equations.

Keywords: Fractional Double Sumudu Transform; Homotopy Analysis Method; Fractional Partial Differential Equations; Caputo Fractional Derivative

1. Introduction

Many physical processes such as diffusion in porous media, heat transport in composite materials, signal loss along a transmission line carry a memory of their own past evolution that integer-order calculus cannot represent. Fractional-order derivatives, developed from Riemann-Liouville through Caputo were built precisely to capture this memory effect and the Caputo formulation has become the working standard in applications because it permits classical, integer-order initial data [1-3]. Most fractional PDEs arising from real models are nonlinear, however their exact solutions are rare, which has pushed research toward semi-analytical series methods such as Adomian decomposition, variational iteration and homotopy-based schemes that avoid discretisation and linearisation alike [4-6]. Among these, Liao's homotopy analysis method (HAM) is distinguished by its auxiliary parameter $\hbar$, which gives explicit control over the convergence of the resulting series [4, 7].

Integral transforms are often coupled to such series methods to simplify the algebra of the fractional derivative term. The Sumudu transform, introduced by Watugala is attractive here since it preserves the units of the original function and reduces cleanly under differentiation [8, 9], motivating hybrid schemes such as the homotopy perturbation Sumudu transform method and the homotopy analysis Sumudu transform method (HASTM), which have solved fractional Burgers', Fisher's and Klein–Gordon type

equations with good agreement against known exact solutions [10-13]. In nearly all of these, however, the transform is applied to only one variable, leaving the spatial operator to be handled term by term within the recursion.

This paper proposes the Fractional Double Sumudu Homotopy Analysis Transform Method (FDSHATM), in which a fractional double Sumudu transform taken jointly in space and time is embedded directly into the homotopy analysis framework, so that the transformed equation is fully explicit before the deformation equations are constructed, while $\hbar$ is retained throughout for convergence control. The method is tested on the time-fractional Burgers' equation and the time-fractional Fisher's equation; in both cases the FDSHATM series is shown to collapse to the known classical solution at $\alpha = 1$, which serves as the basic consistency check adopted here.

2. Preliminaries

In this section, we briefly recall the basic definitions and preliminary results from fractional calculus and integral transform theory that are required for the subsequent analysis.

Definition 2.1 Modified Riemann-Liouville Fractional Derivatives

The modified Riemann–Liouville fractional derivative, proposed by Jumarie [14], is defined as follows.

Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function and let $0<\alpha<1$. Then modified Riemann–Liouville fractional derivative of order α is defined by

$$D_t^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}\left[f(\tau)-f(0)\right]d\tau,\ t>0. \tag{1}$$

If $f'(t)$ exists, the above definition is equivalent to

$$D_t^{\alpha} f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{d}{d\tau}f(\tau)d\tau,\ 0<\alpha<1. \tag{2}$$

Definition 2.2 Sumudu Transform

Over the set of functions, $A=\left\{f(t)\,|\,\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{|t|/\tau_j};t\in(-1)^j\times[0,\infty)\right\}$ the sumudu transform is defined by [15]

$$G(u)=S\left[f(t)\right]=\int_0^{\infty} f(ut)e^{-t}dt;\ u\in(-\tau_1,\tau_2) \tag{3}$$

Definition 2.3 Double Sumudu Transform

Let $f(t,x);t,x\in R_+$ be a function which can be expressed as a convergent infinite series, then, its double Sumudu transform is given by [16],

$$F(u,v)=S_2\left[f(t,x);(u,v)\right]=\frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{t}{v}+\frac{x}{u}\right)}f(t,x)dtdx \tag{4}$$

Definition 2.4 Fractional Sumudu Transform

The Fractional Sumudu transform given by Kılıcman [17] is defined as follows.

Let $f(t)$ be a function defined for $t\geq 0$ and of exponential order for $0<\alpha\leq 1$, then fractional Sumudu transform of order α is defined by,

$$S_\alpha\{f(t)\}(u)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t}\,t^{\alpha-1}f(ut)dt,\ u>0 \tag{5}$$

Definition 2.5 Fractional Double Sumudu Transform

Let $f(x,t)$ denote a function which vanishes for negative values of x and t. Its double Sumudu transform of fractional order or its fractional double Sumudu transform given by Kilicman [18] is defined as

$$S_\alpha^2\left[f(t,x)\right]=G_\alpha^2(u,v)=\int_0^\infty\int_0^\infty E_\alpha\left[-(t+x)^\alpha\right]f(ut,vx)(dt)^\alpha(dx)^\alpha \tag{6}$$

Where $u,v\in C$ and $E_\alpha(x)$ is the Mittag-Leffler function.

Definition 2.6 Fractional Double Sumudu Transform of Fractional Derivative:

Fractional double Sumudu transform of fractional partial derivative respect to t is [18],

$$S_\alpha^2\left\{\partial_t^\alpha f(t,x)\right\}=\frac{G_\alpha^2(u,v)-\Gamma(1+\alpha)G(v)}{u^\alpha} \tag{7}$$

3. Fundamental idea of Fractional Double Sumudu Homotopy Analysis Transform Method (FDSHATM)

To illustrate the basic idea of FDSHATM, consider the following fractional nonlinear partial differential equation,

$$D_t^\alpha u(x,t)+R[u]+N[u]=g(x,t);\ 0<\alpha\le 1 \tag{8}$$

Where D_t^α represents the fractional derivative of $u(x,t)$ in the Caputo's sense,

R and N specifies the linear and nonlinear differential operator, respectively.

$g(x,t)$ represents the source term.

Applying the fractional double Sumudu transform (FDST) with respect to variable x and t on both sides of equation (8), we get

$$S_\alpha^2\left[D_t^\alpha u(x,t)\right]+S_\alpha^2\left[R[u]\right]+S_\alpha^2\left[N[u]\right]=S_\alpha^2\left[g(x,t)\right] \tag{9}$$

Where $S_\alpha^2(\cdot)$ denotes the fractional double Sumudu transform.

Using the differential property of FDST for the caputo derivative D_t^α, equation (9) can be written as

$$\frac{U(u,v)}{u^\alpha}-\frac{\Gamma(1+\alpha)S_x\left[u(x,0)\right]}{u^\alpha}+S_\alpha^2\left[R[u]\right]+S_\alpha^2\left[N[u]\right]=S_\alpha^2\left[g(x,t)\right] \tag{10}$$

On simplifying (10) we get

$$U(u,v)=\Gamma(1+\alpha)S_x\left[u(x,0)\right]+u^\alpha S_\alpha^2\left[g(x,t)\right]-u^\alpha S_\alpha^2\left[R[u]\right]-u^\alpha S_\alpha^2\left[N[u]\right] \tag{11}$$

We now define the nonlinear operator as

$$N\left[\phi(x,t;q)\right]=S_\alpha^2\left[\phi(x,t;q)\right]-\Gamma(1+\alpha)S_x\left[\phi(x,0;q)\right]-u^\alpha S_\alpha^2\left[g(x,t)\right]+u^\alpha S_\alpha^2\left[R[\phi(x,t;q)]\right]+u^\alpha S_\alpha^2\left[N[\phi(x,t;q)]\right] \tag{12}$$

Where $q \in [0,1]$ is the embedding parameter and $\phi(x,t;q)$ is a real function of *x, t* and *q*.

The zero order deformation equation of FDSHATM is constructed as

$$(1-q)S_\alpha^2[\phi(x,t;q)-u_0(x,t)] = \hbar qH(x,t)N[\phi(x,t;q)] \tag{13}$$

Where $\hbar \neq 0$ is the auxiliary parameter,

$H(x,t)$ is the nonzero auxiliary function,

$u_0(x,t)$ is an initial approximation of the solution.

From equation (13) it follows that

$$\phi(x,t;0) = u_0(x,t) \text{ and } \phi(x,t;1) = u(x,t) \tag{14}$$

Thus as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.

Now expanding $\phi(x,t;q)$ in a Taylor's series about $q=0$ we have,

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m \tag{15}$$

$$\text{Where, } u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0} \tag{16}$$

If the auxiliary linear operator, the auxiliary parameter $\hbar$, the auxiliary function $H(x,t)$ and the initial guess $u_0(x,t)$ are properly chosen, then the series given by (15) converges at $q=1$, yielding

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \tag{17}$$

Which represents the solution of the original nonlinear Caputo fractional partial differential equation.

According to the definition of (17), the governing equation can be deduced from the zero-order deformation equation (13).

Define the vector,

$$\vec{u}_m = \{u_0, u_1, u_2, \ldots, u_m\} \tag{18}$$

Differentiating the zero-order deformation equation (13) m-times with respect to q and then dividing by $m!$ and setting $q=0$ we obtain following m^{th} order deformation equation.

$$S_\alpha^2[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(u_{m-1}) \tag{19}$$

$$\text{Where, } \chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{20}$$

$$\text{And } R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{21}$$

Applying the inverse FDST, we finally obtain

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar S_\alpha^{-2}[H(x,t)R_m(u_{m-1})] \tag{22}$$

In this way it is easy to obtain $u_m(x,t)$ for $m \geq 1$.

Then we have,

At m^{th} order choosing proper $\hbar$ we get,

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t) \tag{23}$$

4. Applications

To validate the efficiency and applicability of the proposed algorithm, we present two representative examples.

Example 4.1 Consider the following time fractional Burger's equation [11]

$$D_t^{\alpha} u(x,t) + u(x,t)u_x(x,t) - u_{xx}(x,t) = 0, \quad 0 < \alpha \le 1, \quad t > 0 \tag{24}$$

With the initial condition $u(x,0) = x$.

Solution:

Applying the FDST on both sides of given equation we obtain,

$$\frac{1}{u^{\alpha}} U(u,v) - \frac{\Gamma(1+\alpha)}{u^{\alpha}} S_{\alpha}^{\alpha}[x] + S_{\alpha}^{2}[uu_x] - S_{\alpha}^{2}[u_{xx}] = 0 \tag{25}$$

Therefore $U(u,v) - \Gamma(1+\alpha) S_{\alpha}^{\alpha}[x] - u^{\alpha} S_{\alpha}^{2}[u_{xx} - uu_x] = 0$ (26)

Now define nonlinear operator as

$$N[\phi(x,t;q)] = S_{\alpha}^{2}[\phi(x,t;q)] - \Gamma(1+\alpha) S_{\alpha}^{\alpha}[x] - u^{\alpha} S_{\alpha}^{2}[\phi(x,t;q)_{xx} - \phi(x,t;q)\phi(x,t;q)_x] \tag{27}$$

And thus

$$R_m(\vec{u}_{m-1}) = S_{\alpha}^{2}[u_{m-1}] - (1-\chi_m)\Gamma(1+\alpha) S_{\alpha}^{\alpha}[x] - u^{\alpha} S_{\alpha}^{2}\left[(u_{m-1})_{xx} - \sum_{k=0}^{m-1} u_k (u_{m-1-k})_x\right] \tag{28}$$

The m^{th} order deformation equation is given by

$$S_{\alpha}^{2}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}), \; m \ge 1 \tag{29}$$

Applying the inverse FDST we obtain

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar S_{\alpha}^{-2}[R_m(\vec{u}_{m-1})] \tag{30}$$

Now solving the above equation for $m = 1,2,\ldots$, we obtain

$$u_1(x,t) = \frac{\hbar x t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \hbar(1+\hbar)\frac{x t^{\alpha}}{\Gamma(\alpha+1)} + 2\hbar^2 \frac{x t^{2\alpha}}{\Gamma(2\alpha+1)}, \ldots$$

Substituting $u_0, u_1, u_2, \ldots$ and choosing $\hbar = -1$ we get the FDSHATM solution as,

$$u(x,t) = x - \frac{t^{\alpha} x}{\Gamma(\alpha+1)} + \frac{2t^{2\alpha} x}{\Gamma(2\alpha+1)} - \ldots = x\left(1 + \sum_{m=1}^{\infty} \frac{m!(-1)^m t^{\alpha m}}{\Gamma(\alpha m+1)}\right). \tag{31}$$

The exact result for given equation when $\alpha = 1$ is $\frac{x}{1-t}$.

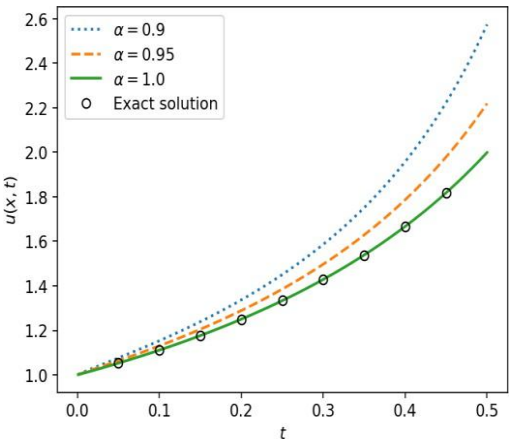


Figure 1. Comparison of the FDSHATM solution $u(1,t)$ for α = 0.9, 0.95, 1.0 with the exact solution of eq. (24)

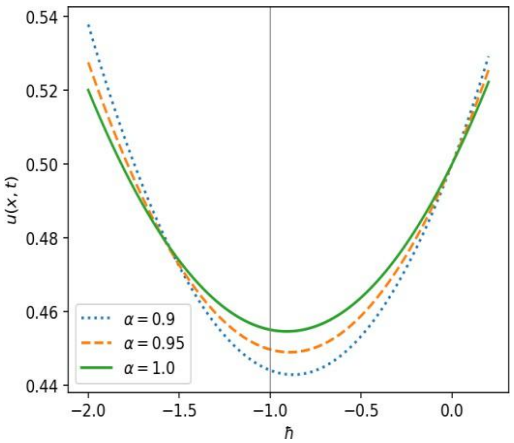


Figure 2. *fl* -curve of the FDSHATM solution at $x = 0.5$, $t = 0.1$ for $\alpha = 0.9, 0.95, 1.0$ for eq. (24)

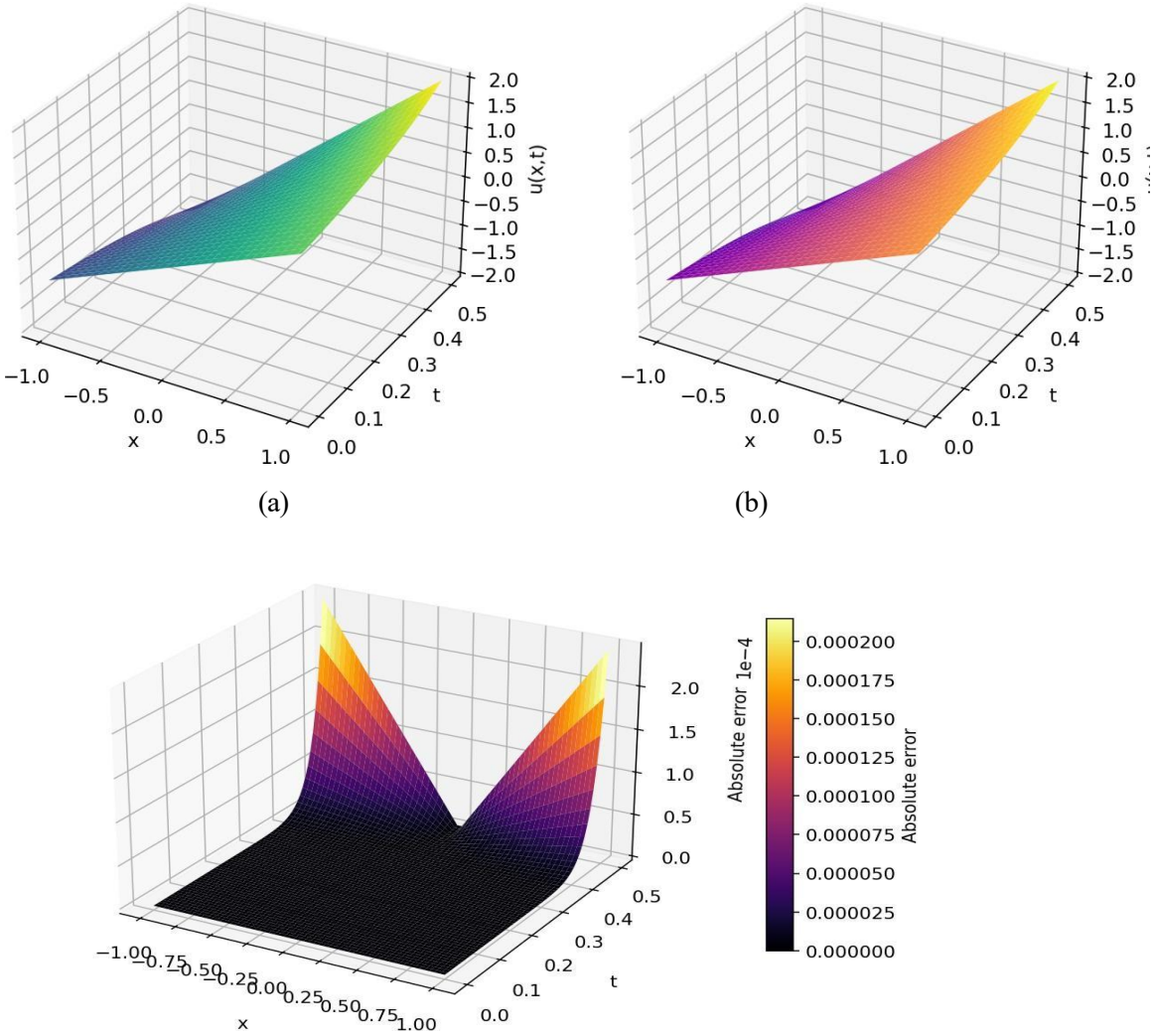


Figure 3. Three-dimensional plots of (a) the FDSHATM solution and (b) the exact solution over the (x, t) domain at α = 1, and (c) the Absolute error surface |u_exact − u_FDSHATM| over the (x, t) domain at $\alpha = 1$ for eq. (24)

Example 4.2 Consider the non-linear time fractional Fisher's equation [12]

$$D_t^{\alpha}u(x,t) = u_{xx}(x,t) + 6u(x,t)\left(1-u(x,t)\right);\ 0<\alpha\le 1 \tag{32}$$

with initial condition $u(x,0) = \dfrac{1}{\left(1+e^{x}\right)^{2}}$. (33)

Solution: Applying the FDST on both sides of given equation we have,

$$U(u,v) = \Gamma(1+\alpha)S_{\alpha}^{x}\left[u_0\right] + u^{\alpha}\left[S_{\alpha}^{2}\left[u_{xx}\right] + 6S_{\alpha}^{2}\left[u\right] - 6S_{\alpha}^{2}\left[u^{2}\right]\right] \tag{34}$$

Now define nonlinear operator as

$$N\left[\phi(x,t;q)\right] = S_{\alpha}^{2}\left[\phi(x,t;q)\right] - \Gamma(1+\alpha)S_{x}^{\alpha}\left[u_0\right] - u^{\alpha}S_{\alpha}^{2}\left[\phi(x,t;q)_{xx} + 6\phi(x,t;q) - 6\phi^{2}(x,t;q)\right] \tag{35}$$

And thus

$$R_m(\vec{u}_{m-1}) = S_{\alpha}^{2}\left[u_{m-1}\right] - (1-\chi_m)\Gamma(1+\alpha)S_{x}^{\alpha}\left[u_0\right] - u^{\alpha}S_{\alpha}^{2}\left[(u_{m-1})_{xx} + 6u_{m-1} - 6\sum_{k=0}^{m-1}u_k u_{m-1-k}\right] \tag{36}$$

Applying the inverse FDST on above equation we get,

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar S_{\alpha}^{-2}\left[R_m(\vec{u}_{m-1})\right] \tag{37}$$

Now solving the above equation for $m = 1,2,\ldots$, we obtain

$$u_1(x,t) = \frac{-10\hbar e^{x}}{(1+e^{x})^{3}}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \frac{-10\hbar(1+\hbar)e^{x}}{(1+e^{x})^{3}}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{50\hbar^{2}(-1+2e^{x})e^{x}}{(1+e^{x})^{4}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\ldots$$

Substituting $u_0, u_1, u_2, \ldots$ and choosing $\hbar = -1$ we get the FDSHATM solution as,

$$u(x,t) = \frac{1}{\left(1+e^{x}\right)^{2}} + 10\frac{e^{x}}{\left(1+e^{x}\right)^{3}}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 50\frac{e^{x}\left(-1+2e^{x}\right)}{\left(1+e^{x}\right)^{4}}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots \tag{38}$$

The closed form solution corresponding to series is,

$$u(x,t) = \frac{1}{\left(1+e^{x-\frac{5t^{\alpha}}{\Gamma(\alpha+1)}}\right)^{2}}. \tag{39}$$

The exact result for given equation when $\alpha = 1$ is $u(x,t) = \dfrac{1}{\left(1+e^{x-5t}\right)^{2}}$, which is the classical Fisher's equation solution.

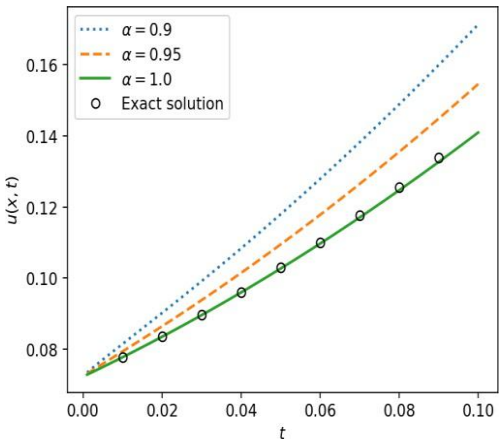


Figure 4. Comparison of the FDSHATM solution $u(1,t)$ for α = 0.9, 0.95, 1.0 with the exact solution of eq. (32)

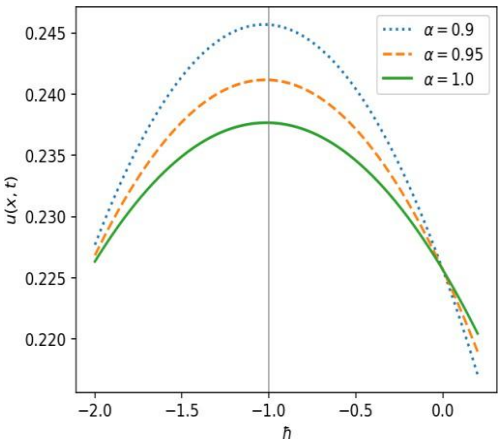


Figure 5. *fl* -curve of the FDSHATM solution at $x = 0.1$, $t = 0.01$ for $\alpha = 0.9, 0.95, 1.0$ of eq. (32)

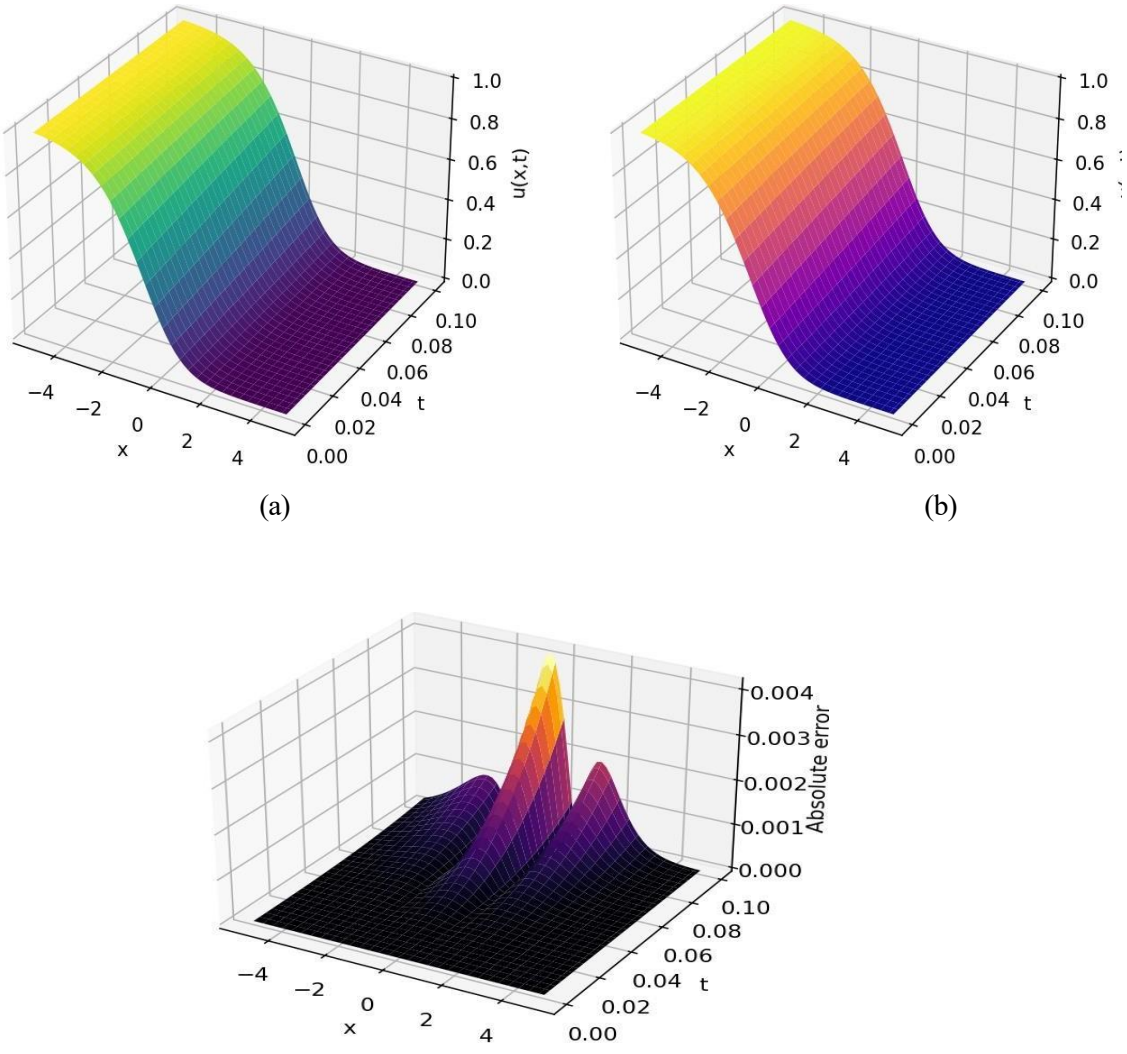


Figure 6. Three-dimensional plots of (a) the FDSHATM approximate solution and (b) the exact solution over the (x, t) domain at $\alpha = 1$ and (c) Absolute error surface $|u_exact - u_FDSHATM|$ over the (x, t) domain at $\alpha = 1$ for eq. (32)

5. Results and Discussion

To validate the proposed Fractional Double Sumudu Homotopy Analysis Transform Method (FDSHATM), two nonlinear fractional PDEs - the time-fractional Burgers' equation (24) and the time-fractional Fisher's equation (32) were solved and compared against their known exact solutions at $\alpha = 1$.

Figure 1 compares $u(1, t)$ for $\alpha = 0.9, 0.95, 1.0$ with the exact solution $\alpha \to 1$ the FDSHATM curves converge to the exact solution of example 4.1, while lower α values reveal the memory effect of the fractional derivative. Figure 2 shows the $\hbar$ - curve at $x = 0.5$, $t = 0.1$; its flat plateau justifies the standard choice $\hbar = -1$ used in the computation. Figure 3 (a) shows the FDSHATM and (b) shows the exact solution surfaces at $\alpha = 1$ in close agreement over the full (x, t) domain. Figure 3 (c) gives the absolute error surface, which remains small and grows only gradually with t, confirming the accuracy of the truncated series.

A parallel analysis was performed against the exact solution for example 4.2. Figure 4 shows convergence of the FDSHATM solution to the exact wave as $\alpha \to 1$, with lower α altering the propagation profile. Figure 5 shows $\hbar$ -curve again validates $\hbar = -1$ as lying within the convergence region. Figure 6 confirms that the characteristic wave front shape is well reproduced by FDSHATM over the domain. In Figure 6 (c), error surface shows slightly higher error near the steep wave front, a known limitation of series methods near sharp gradients.

6. Conclusion

This paper introduced the Fractional Double Sumudu Homotopy Analysis Transform Method (FDSHATM), embedding a double Sumudu transform taken jointly in space and time directly into the homotopy analysis framework rather than transforming a single variable as in earlier hybrid schemes. Specialising the general m^{th}-order deformation equation to the time-fractional Burgers' and Fisher's equations gives a recursion in which each term follows from a single inverse transform, free of discretisation or linearisation. The graphs for several values of α show the solution surface deforming smoothly and approaching the classical profile as $\alpha \to 1$, consistent with the closed-form check obtained at $\alpha = 1$, $\hbar = -1$. Since $\hbar$ is retained throughout, the method keeps the usual freedom to adjust convergence for less cooperative equations, supporting FDSHATM as a workable route to series solutions of nonlinear fractional PDEs.

REFERENCES

[1] I. Podlubny, “Fractional Differential Equations”, Academic Press, San Diego, (1999).
[2] M. Caputo, “Linear Models of Dissipation whose Q is almost Frequency Independent-II”, Geophys. J. R. Astron. Soc., vol. 13, no. 5, (1967), pp. 529-539.
[3] K. S. Miller and B. Ross, “An Introduction to the Fractional Calculus and Fractional Differential Equations”, Wiley, New York, (1993).
[4] S. J. Liao, “Homotopy Analysis Method: A New Analytic Method for Nonlinear Problems”, Appl. Math. Mech. (English Edition), vol. 19, no. 10, (1998), pp. 957-962.
[5] A. M. Wazwaz and A. Gorguis, “An analytic study of Fisher’s equation by using Adomian decomposition method”, Appl. Math. Comput., vol. 154, no. 3, (2004), pp. 609-620.
[6] H. Jafari and H. K. Jassim, “Local Fractional Variational Iteration Method for Solving Nonlinear Partial Differential Equations within Local Fractional Operators”, Appl. Appl. Math., vol. 10, no. 2, (2015), pp. 1055-1065.
[7] S. J. Liao, “Beyond Perturbation: Introduction to the Homotopy Analysis Method”, Chapman & Hall/CRC, Boca Raton, (2004).
[8] G. K. Watugala, “Sumudu Transform: A New Integral Transform to Solve Differential Equations and Control Engineering Problems”, Int. J. Math. Educ. Sci. Technol., vol. 24, no. 1, (1993), pp. 35-43.

[9] K. Wang and S. Liu, “A New Sumudu Transform Iterative Method for Time-Fractional Cauchy Reaction-Diffusion Equation”, SpringerPlus, vol. 5, (2016), p. 865.
[10] A. S. Abdel-Rady, S. Z. Rida, A. A. M. Arafa, and H. R. Abdel-Rahim, “Approximate Analytical Solutions of the Fractional Nonlinear Dispersive Equations Using Homotopy Perturbation Sumudu Transform Method (HPSTM)”, Int. J. Innov. Sci. Eng. Technol., vol. 1, no. 9, (2014), pp. 257-267.
[11] H. K. Jassim, H. Ahmad, A. Shamaoon, and C. Cesarano, “An Efficient Hybrid Technique for the Solution of Fractional-Order Partial Differential Equations”, Carpathian Math. Publ., vol. 13, no. 3, (2021), pp. 790-804.
[12] P. Veeresha, D. G. Prakasha, and H. M. Baskonus, “Novel Simulations to the Time-Fractional Fisher’s Equation”, Math. Sci., vol. 13, no. 1, (2019), pp. 33-42.
[13] M. Kurulay, “Solving the Fractional Nonlinear Klein-Gordon Equation by Means of the Homotopy Analysis Method”, Adv. Differ. Equ., vol. 2012, (2012), Article 187.
[14] G. Jumarie, “Modified Riemann-Liouville Derivative and Fractional Taylor Series of Nondifferentiable Functions: Further Results”, Comput. Math. Appl., vol. 51, nos. 9-10, (2006), pp. 1367-1376.
[15] G. K. Watugala, “Sumudu Transform: A New Integral Transform to Solve Differential Equations and Control Engineering Problems”, Int. J. Math. Educ. Sci. Technol., vol. 24, no. 1, (1993), pp. 35-43.
[16] F. Jarad and K. Tas, “Application of Sumudu and Double Sumudu Transforms to Caputo-Fractional Differential Equations”, J. Comput. Anal. Appl., vol. 14, no. 3, (2012), pp. 475-483.
[17] A. Kılıçman and H. E. Gadain, “On the Fractional Sumudu Transform”, Appl. Math. Comput., vol. 217, (2010), pp. 1937-1945.
[18] A. Kılıçman and O. Altun, “Some Remarks on the Fractional Sumudu Transform and Applications”, Appl. Math. Inf. Sci., vol. 8, no. 6, (2014), pp. 2881-2888.

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