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Module Amenability of Banach Algebras on Some Modules

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ABSTRACT. In this paper, we investigate some properties of modular amenability of Banach algebras. Also, we will showd that If A be a Banach algebra and A-module two-side Banach algebra, then the cyclic modular amenability A and $A^\#$ is investigated.

1. INTRODUCTION

The concept of homology was considered one of the most important topics that studied in the middle of the 20th century in algebra trends, including algebraic geometry, algebraic topology, and displacement algebra, until Grothendieck, Kamowitz, and then Johnson raised this topic and used it in Banach algebras, and related the concept of cohomology with the concept of homology by an analogy. Moreover Amini in [1] developed the concept of module amenability for a class of Banach algebras which is in fact a generalization of the Johnson's amenability. Also the author of the present paper along with Miri and Nasrabadi in [6] studied concept of cyclic module amenability of Banach algebras.

Let A be a Banach algebra, and X be a Banach A-bimodule. A bounded linear map $D : A \longrightarrow X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

For each $x \in X$, $\mathrm{ad}_x : A \longrightarrow X$ defined with

$$\mathrm{ad}_x(a) = a \cdot x - x \cdot a \qquad (a \in A),$$

is derivation. Derivations of this form are called inner derivations. Also, a Banach algebra A is called amenable if for any Banach A-bimodule X, every derivation $D : A \longrightarrow X^*$ is inner, where X^* is the dual of X. Also, A is called weak amenable if every derivation $D : A \longrightarrow A^*$ is inner. A derivation $D : A \longrightarrow A^*$ is called cyclic if

$$[D(a)](b) + [D(b)](a) = 0 \qquad (a, b \in A).$$

A Banach algebra A is called cyclic amenable if every cyclic derivation $D : A \longrightarrow A^*$ is inner.

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The concepts of module amenability and weak module amenability for Banach algebras which is Banach module over another Banach algebra, has been developed by Amini in [**?**] and Amini along with Bagha in [**?**], respectively.

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, and let X be a left Banach A-module and a Banach $\mathfrak{A}$-bimodule with the following compatible actions is called left Banach A-$\mathfrak{A}$-module:

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \;\; a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \;\; a \cdot (x \cdot \alpha) = (a \cdot x) \cdot \alpha,$$

for all $a \in A$, $\alpha \in \mathfrak{A}$ and $x \in X$. The right Banach A-$\mathfrak{A}$-module is defined semilarly. If X be two-side Banach A-$\mathfrak{A}$-module, it is called Banach A-$\mathfrak{A}$-module. Also, X is called a commutative (bi-commutative) Banach A-$\mathfrak{A}$-module, if $\alpha \cdot x = x \cdot \alpha (a.x = x.a)$ for all $\alpha \in \mathfrak{A}$, $a \in A$ and $x \in X$. If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined as usual:

$$\langle f \cdot \alpha, x\rangle = \langle f, \alpha \cdot x\rangle, \quad \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle,$$
$$\langle \alpha \cdot f, x\rangle = \langle f, x \cdot \alpha\rangle, \quad \langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle,$$
$$(a \in A, \alpha \in \mathfrak{A}, x \in X, f \in X^*).$$

Notice that, if A is a commutative Banach $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A^* is also a commutative Banach A-$\mathfrak{A}$-module.

A bounded map $D : A \longrightarrow X$ is called a $\mathfrak{A}$-module derivation if for all $a, b \in A$ and $\alpha \in \mathfrak{A}$:

$$D(a \pm b) = D(a) \pm D(b), D(\alpha \cdot a) = \alpha \cdot D(a), D(a \cdot \alpha) = D(a) \cdot \alpha,$$

and

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

Moreover, $\mathfrak{A}$-module derivation $D : A \longrightarrow A^*$ is called cyclic if it satisfy the following condition $[D(a)](b) + [D(b)](a) = 0$.

Definition 1.1. A Bananch algebra A is called $\mathfrak{A}$-module amenable if for any commutative Banach A-$\mathfrak{A}$-module X, each $\mathfrak{A}$-module derivation $D : A \longrightarrow X^*$ is inner. A commutative Bananch $\mathfrak{A}$-bimodule A is called weak $\mathfrak{A}$-module amenable, if every $\mathfrak{A}$-module derivation $D : A \longrightarrow A^*$ is inner.

Theorem 1.2. *Let A and B be Banach algebras and commutative Banach $\mathfrak{A}$-bimodule. Suppose A is cyclic $\mathfrak{A}$-module amenable. If $\phi : A \longrightarrow B$ and $\psi : A \longrightarrow B$ are continuous $\mathfrak{A}$-module homomorphisms such that $\psi \circ \phi = I_{\mathrm{B}}$, then B is cyclic $\mathfrak{A}$-module amenable.*

Proof. Suppose that, $D : B \longrightarrow B^*$ be a cyclic $\mathfrak{A}$-module derivation. We consider $\tilde{D} = \varphi^* \circ D \circ \varphi : A \longrightarrow A^*$. Then, for any $a_1, a_2, a_3 \in A$ we have;

$$\begin{aligned}
\langle a_3, \tilde{D}(a_1a_2)\rangle &= \langle \varphi(a_3), D(\varphi(a_1)\varphi(a_2))\rangle \\
&= \langle \varphi(a_3), D(\varphi(a_1)) \cdot \varphi(a_2) + \varphi(a_1) \cdot D(\varphi(a_2))\rangle \\
&= \langle \varphi(a_2a_3), D(\varphi(a_1))\rangle + \langle \varphi(a_3a_1), D(\varphi(a_2))\rangle \\
&= \langle a_2a_3, \tilde{D}(a_1)\rangle + \langle a_3a_1, \tilde{D}(a_2)\rangle \\
&= \langle a_3, \tilde{D}(a_1) \cdot a_2 + a_1 \cdot \tilde{D}(a_2)\rangle.
\end{aligned}$$

Hence, $\tilde{D}$ is a continous derivation. Moreover, $\tilde{D}$ is cyclic, since D is cyclic. Now, let $\alpha \in \mathfrak{A}$ and $a_1, a_2 \in A$,

$$\begin{aligned}
\langle a_2, \tilde{D}(\alpha \cdot a_1)\rangle &= \langle \varphi(a_2), D(\varphi(\alpha \cdot a_1))\rangle \\
&= \langle \varphi(a_2), \alpha \cdot D(\varphi(a_1))\rangle \\
&= \langle \varphi(a_2 \cdot \alpha), D(\varphi(a_1))\rangle \\
&= \langle a_2 \cdot \alpha, \tilde{D}(a_1)\rangle \\
&= \langle a_2, \alpha \cdot \tilde{D}(a_1)\rangle,
\end{aligned}$$

Therefore,

$$\tilde{D}(\alpha \cdot a_1) = \alpha \cdot \tilde{D}(a_1)$$

and semilarlly

$$\tilde{D}(a_1 \cdot \alpha) = \tilde{D}(a_1) \cdot \alpha.$$

Thus, $\tilde{D}$ is $\mathfrak{A}$-module derivation. Now, since A is $\mathfrak{A}$-module cyclic ameanable. Then, there exist $f \in A^*$ such that $\tilde{D} = ad_f$. By $\varphi \circ \psi = I_B$ conclude that $\psi^* \circ \varphi^* = I_{B^*}$.

For any $b_1, b_2 \in B$ we get

$$\begin{aligned}
\langle b_2, D(b_1)\rangle &= \langle b_2, \psi^* \circ \varphi^* \circ D \circ \varphi \circ \psi(b_1)\rangle \\
&= \langle b_2, \psi^*(\tilde{D}(\psi(b_1)))\rangle \\
&= \langle \psi(b_2), ad_f(\psi(b_1))\rangle \\
&= \langle \psi(b_2), \psi(b_1) \cdot f - f \cdot \psi(b_1)\rangle \\
&= \langle \psi(b_2b_1 - b_1b_2), f\rangle \\
&= \langle b_2b_1 - b_1b_2, \psi^*(f)\rangle \\
&= \langle b_2, b_1 \cdot \psi^*(f) - \psi^*(f) \cdot b_1\rangle \\
&= \langle b_2, ad_{\psi^*(f)}(b_1)\rangle.
\end{aligned}$$

which is show that

$D = ad_{\psi^*(f)}$. So, B is cyclic $\mathfrak{A}$-module amenable. □

Theorem 1.3. *Let A be an non-unital Banach algebra, then A is cyclic amenable if and only if $A^{\#} := A \bigoplus \mathbb{C}$ is too.*

Proof. Firstly, suppose that $D : A^{\#} \longrightarrow (A^{\#})^*$ be an inner derivation, we define

$$\tilde{D} : A^{\#} \longrightarrow (A^{\#})^*$$
$$(a, \alpha) \longrightarrow \tilde{D}(a, \alpha).$$

where $\tilde{D}(a, \alpha) := (D(a), 0)$.
so has the form (f_0, λ) which $f_{0a} \in A^*$. Due to $D(a) \in A^*$, so we give f_0 and $D(a)$ as the same and $\lambda = 0$. Now, at the first $\tilde{D}$ is a derivation, becuse $a, b \in A$ and *andequvalentlyforany*$\alpha, \beta \in \mathbb{C}$ and equvalently for any $(b, \beta), (a, \alpha) \in A^{\#}$ we have

$$\begin{aligned}
\tilde{D}((a, \alpha).(b, \beta)) &= \tilde{D}(ab + a\beta + \alpha b, \alpha\beta)\\
&= (D(ab + a\beta + \alpha b), 0)\\
&= D(ab + a\beta + \alpha b, 0)\\
&= D(ab + a\beta + \alpha b)\\
&= D(ab) + D(a\beta) + D(\alpha b)\\
&= a.(Db) + (Da).b + \beta D(a) + \alpha D(b)\\
&= a.(Db) + (Da).b + \beta D(a) + \alpha D(b) + (D(a))(b) + (D(b))(a)\\
&= (Da).b + \beta D(a) + (D(a))(b) + a.(Db) + \alpha D(b) + (D(b))(a)\\
&= (D(a), 0).(b, \beta) + (a, \alpha).(D(b), 0)\\
&= \tilde{D}(a, \alpha).(b, \beta) + (a, \alpha).\tilde{D}(b, \beta)
\end{aligned}$$

At the second, based on the definition of $\tilde{D}$ it is clear that $\tilde{D}$ is cyclic.

□

Definition 1.4. Let A be Banach Algebra and $\mathrm{A}^{\#}$-module two-side Banach with compatible action. If

$\mathrm{B} = (\mathrm{A} \bigoplus \mathrm{A}^{\#}, \bullet)$ with $\bullet$ as follow;

$$(a, u) \bullet (b, v) = (ab + av + u, uv) \qquad , \qquad (a, b \in \mathrm{A}, u, v \in \mathrm{A}^{\#}).$$

Also, let $\mathrm{A}^{\#}$ on B be define as below;

$$u.(a, v) = (u.a, uv), \qquad (a, v).u = (a.u, vu), \qquad , (a \in A, u, v \in \mathrm{A}^{\#}).$$

then B is an unital Banach algebra and $\mathrm{A}^{\#}$ -module two-side Banach algebra.

Theorem 1.5. *Let A be Banach algebra A -module two-side Banach with compatible action. Let B be Banach algebra which is defined at 1.4,* *then* A *is an* $-\mathrm{A}^{\#}$ *amenable* ω^{*} *-modular approximate if and only if* B *be an* $-\mathrm{A}^{\#}$ *amenable* ω^{*} *-modular approximate.*

Proof. At first let X be B -$\mathrm{A}^{\#}$ -module commutative Banach and $D : \mathrm{B} \longrightarrow X^{*}$ be $\mathrm{A}^{\#}$-modular derevation. $u \in \mathrm{A}^{\#}$ we have.

$$D(u) = uD(1) = 0$$

so D to modular derevation $D : \mathrm{A} \longrightarrow X^{*}$ is restricted. Since, X is $\mathrm{A}^{\#} - \mathrm{A}$-module commutative Banach. Thus, D is inner. Conversly, let X

$\mathrm{A}^{\#} - \mathrm{A}$-module commutative Banach. we can see that X is an $\mathrm{A}^{\#} - \mathrm{B}$ -module commutative Banach. Now, $\mathrm{A}^{\#}$-module $D : \mathrm{A} \longrightarrow X^{*}$ extentable to $\mathrm{A}^{\#}$-module derevation $\tilde{D} : \mathrm{B} \longrightarrow X^{*}$. we have;

$$\tilde{D}(a, u) = D(a) \quad , \quad (\forall a \in \mathrm{A} \ , \ u \in \mathrm{A}^{\#})$$

which is $\tilde{D}$ is inner, so D is inner. Therefor, A is an $\mathrm{A}^{\#}$-modular amenablity. □

Theorem 1.6. *Let* A *be Banach algebra* A*-module two-side Banach with compatible action and* B *Banach algebra which is defined at 1.4. Also, let* A *be a left and right essential* A *is an* $-\mathrm{A}^{\#}$*- modular approximate amenable* ω^{*} *if and only if* A *is* A*-* ω^{*} *- modular approximate amenable.*

Proof. First side is evidence. because, $\mathrm{A}^{\#}$ -module is A -module. Conversly, let X be $\mathrm{A}^{\#}$ $-\mathrm{A}$-module Banach and $D : \mathrm{A} \longrightarrow X^{*}$ be derevation A-module. Then, X is $\mathrm{A}^{\#} - \mathrm{A}$-module commutative Banach. Therefor, A is -module left (right) essential. So D is linear. Then,

$$D((\alpha, \lambda).a) = D(\alpha.a+\lambda a) = D(\alpha.a)+D(\lambda a) = \alpha.D(a)+\lambda D(a) = (\alpha, \lambda).D(a) \ (\forall \lambda \in, \alpha \in \mathrm{A}, a \in \mathrm{A}).$$

semilarly, D is homomorphisem. $\mathrm{A}^{\#}$-right modular. Then, D is derevation. $\mathrm{A}^{\#}$-modular and so is inner. Hence, A is A-modular amenable. □

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Data availability

No data was used for the research described in the article.

Disclosure statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

References

[1] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341–366.

[2] D. Gabor, *Theory of communications*, J. Inst. Elect. Eng., **93** (1946), 429–457.

[3] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.

[4] M. H. Rezaei gol and E. Nasrabadi, *On Cyclic Amenability of Triangular Banach Algebras*, Neuroquantology, (2024), 22(5), 650–655. DOI: 10.48047/nq.2024.22.5.nq25065

[5] F. Ghahramani, R. J. Loy, and Y. Zhang, *Generalized notions of amenability II*, J. Funct. Anal. 254, 1776–1810, 2008.

[6] G. H. Esslamzadeh, and B. Shojaee, *Approximate weak amenability of Banach algebras*, Bull. Belg. Math. Soc. Simon Stevin. 18 , 415–429, 2011.

[7] M. R. Miri, E. Nasrabadi, and M. H Rezaei Gol, *Cyclic Module Amenability of Banach Algebras*, International Journal of Pure and Applied Mathematics (120)3, 315–327, 2018.

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