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Fixed Point Results for α -Admissible Mappings in Rectangular Metric Spaces

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Abstract: In this paper, we shall prove the fixed point theorems in rectangular metric space for generalized contractions using α -admissible mappings. In the end, we shall discuss about consequences of our main results.

Keywords: α -admissible mappings, complete rectangular metric space and fixed point.

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1. Introduction: In 1922, Banach gave a principle to obtain the fixed point in the complete metric space. Since then, many researchers have worked on the Banach fixed point theorem (see [1-9], [11-22]) and tried to generalize this principle. In 2012, Samet *et al.* [23] introduced the new concepts of mappings called α -admissible mappings in metric space. Recently, in 2013 Farhan *et al.* [2] gave new contractions using α -admissible mapping in metric spaces.

In this paper, we shall generalize Farhan's *et al.* [2] contractions and give fixed point theorems for such contractions.

2. Preliminaries: To prove our main results we need some basic definitions from literature as follows:

Definition 2.1. [10] Let X be a set. A rectangular metric space (RMS) is an ordered pair (X, d) where d is a function $d : X \times X \rightarrow \mathbb{R}$ such that

- (1) $(x, y) \geq 0$,
- (2) $(x, y) = 0$ iff $x = y$,
- (3) $(x, y) = d(y, x)$,
- (4) $(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

For all $x, y, u, v \in X$.

Definition 2.2. [10] A sequence $\{x_n\}$ in RMS (X, d) is said to converge if there is a point $x \in X$ and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for every $n > N$.

Definition 2.3. [10] A sequence $\{x_n\}$ in a RMS (X, d) is Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $(x_n, x_m) < \epsilon$ for every $n, m > N$.

Definition 2.4. [10] RMS (X, d) is said to be complete if every Cauchy sequence is convergent.

Definition 2.5. [23] Let $f: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$. We say that f is an α -admissible mapping if

$$(x, y) \geq 1 \text{ implies } \alpha(fx, fy) \geq 1, \quad x, y \in X.$$

3. Main Results:

Theorem 3.1. Let (X, d) be a complete RMS and $T: X \rightarrow X$ be an α -admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(d(Tx, Ty) + l)^{\alpha(x, Tx)\alpha(y, Ty)} \leq \beta(M(x, y))M(x, y) + l, \quad \forall x, y \in X \text{ and } l \geq 1. \quad (3.1)$$

$$\text{Where } M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx).d(Ty, y)}{d(x, y)}, \frac{d(x, Tx)(1+d(Ty, y))}{1+d(x, y)} \right\}$$

Suppose that if T is continuous and

If there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$. Construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$, $\forall n \in \mathbb{N}$.

If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then $Tx_n = x_n$ and we are done.

So, we suppose that $(x_n, x_{n+1}) > 0$, $\forall n \in \mathbb{N}$.

Since T is α -admissible, there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$ which implies $(x_0, x_1) \geq 1$.

Similarly, we can say that $(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$.

By continuing this process, we get

$$(x_n, x_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}. \quad (3.2)$$

By using equation (3.2), we have

$$d(x_n, x_{n+1}) + l = d(Tx_{n-1}, Tx_n) + l \leq (d(Tx_{n-1}, Tx_n) + l)^{\alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Tx_n)}.$$

Now using equation (3.1), we get

$$d(x_n, x_{n+1}) + l \leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) + l, \quad (3.3)$$

$$(x_{n-1}, x_n) = \max \left\{ d(x_{n-1}, x_n), (x_{n-1}, Tx_{n-1}), (x_n, Tx_n), \frac{d(x_{n-1}, Tx_{n-1}).d(Tx_n, x_n)}{d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, Tx_{n-1})(1+d(Tx_n, x_n))}{1+d(x_{n-1}, x_n)} \right\}$$

$$= \max \{(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

Assume that if possible $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$.

Then, $(x_{n-1}, x_n) = d(x_n, x_{n+1})$. Using

this in equation (3.3), we get

$$(x_n, x_{n+1}) < \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \quad (3.4)$$

$\Rightarrow (x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction. So

$$(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \forall n.$$

It follows that the sequence $\{(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive real numbers. So, it is convergent and suppose that $\lim_{n \rightarrow \infty} (x_n, x_{n+1}) = d$. Clearly, $d \geq 0$.

Claim: $d = 0$.

Equation (3.4) implies that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq (d(x_{n-1}, x_n)) \leq 1,$$

Which implies that $\lim_{n \rightarrow \infty} (d(x_{n-1}, x_n)) = 1$.

Using the property of the function β , we conclude that

$$\lim_{n \rightarrow \infty} (x_n, x_{n+1}) = 0. \quad (3.5)$$

In the similar way, we can prove that

$$\lim_{n \rightarrow \infty} (x_n, x_{n+2}) = 0. \quad (3.6)$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences (k) and $n(k)$ such that for all positive integers k , we have

$$n(k) > m(k) > k, d(x_{n(k)}, x_{m(k)}) \geq \epsilon \text{ and } d(x_{n(k)}, x_{m(k)-1}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)+1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \epsilon + d(x_{m(k)-1}, x_{m(k)+1}) + d(x_{m(k)-1}, x_{m(k)}), \end{aligned}$$

for all $k \in \mathbb{N}$.

Taking the limit as $k \rightarrow +\infty$ in the above inequality and using equations (3.5) and (3.6), we get

$$\lim_{k \rightarrow +\infty} (x_{n(k)}, x_{m(k)}) = \epsilon. \quad (3.7)$$

Again, by triangle inequality, we have

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) - d(x_{m(k)-1}, x_{m(k)}) - d(x_{n(k)-1}, x_{n(k)}) &\leq d(x_{n(k)-1}, x_{m(k)-1}) \\ d(x_{n(k)-1}, x_{m(k)-1}) &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{n(k)}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)}). \end{aligned}$$

Taking the limit as $k \rightarrow +\infty$, together with (3.5) - (3.7), we deduce that

$$\lim_{k \rightarrow +\infty} (x_{n(k)-1}, x_{m(k)-1}) = \epsilon. \quad (3.8)$$

From equations (3.1), (3.2), (3.6) and (3.8), we get

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) + l &\leq (d(x_{n(k)}, x_{m(k)}) + l)^{\alpha(x_{n(k)-1}, Tx_{n(k)-1})\alpha(x_{m(k)-1}, Tx_{m(k)-1})}, \\ &= (d(Tx_{n(k)-1}, Tx_{m(k)-1}) + l)^{\alpha(x_{n(k)-1}, Tx_{n(k)-1})\alpha(x_{m(k)-1}, Tx_{m(k)-1})} \\ &\leq (M(x_{n(k)-1}, x_{m(k)-1})M(x_{n(k)-1}, x_{m(k)-1}) + l) \end{aligned} \quad (3.9)$$

$$\begin{aligned} M(x_{n(k)-1}, x_{m(k)-1}) &= \max \{d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), \\ &\frac{d(x_{n(k)-1}, Tx_{n(k)-1}) \cdot d(Tx_{m(k)-1}, x_{m(k)-1})}{d(x_{n(k)-1}, x_{m(k)-1})}, \frac{d(x_{n(k)-1}, Tx_{n(k)-1})(1+d(Tx_{m(k)-1}, x_{m(k)-1}))}{1+d(x_{n(k)-1}, x_{m(k)-1})}\}, \\ &= \max \{(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), \\ &\frac{d(x_{n(k)-1}, x_{n(k)}) \cdot d(x_{m(k)-1}, x_{m(k)})}{d(x_{n(k)-1}, x_{m(k)-1})}, \frac{d(x_{n(k)-1}, x_{n(k)})(1+d(x_{m(k)-1}, x_{m(k)}))}{1+d(x_{n(k)-1}, x_{m(k)-1})}\}. \end{aligned}$$

Taking $k \rightarrow \infty$, we have

$$(x_{n(k)-1}, x_{m(k)-1}) = \max \{\epsilon, 0, 0, 0, 0\}. \text{So,}$$

equation (3.9) implies that

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq \beta(M(x_{n(k)}, x_{m(k)})M(x_{n(k)}, x_{m(k)})) \leq 1,$$

Letting $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} (d(x_{n(k)}, x_{m(k)}) = 1.$$

By using definition of β function, we get

$$\Rightarrow \lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}) = 0 < \epsilon, \text{ which is a contradiction.}$$

Hence, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete space, so $\{x_n\}$ is convergent and assume that $x_n \rightarrow x$ as $n \rightarrow \infty$.

Since T is continuous, then we have

$$Tx = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

So, x is a fixed point of T .

Theorem 3.2. Assume that all the hypothesis of Theorem 3.1 hold. Adding the following condition:

$$\text{If } x = Tx, \text{ then } (x, Tx) \geq 1.$$

We obtain the uniqueness of fixed point.

Proof: Let z and z^* be two distinct fixed point of T in the setting of Theorem 3.1 and above defined condition holds, then

$$(z, Tz) \geq 1 \text{ and } \alpha(z^*, Tz^*) \geq 1.$$

$$\text{So, } d(Tz, Tz^*) + l \leq (d(Tz, Tz^*) + l)^{\alpha(z, Tz)\alpha(z^*, Tz^*)}$$

$$\leq \beta(M(z, z^*))M(z, z^*) + l. \quad (3.10)$$

$$\begin{aligned} \text{Where } M(z, z^*) &= \max \left\{ d(z, z^*), d(Tz, z), d(Tz^*, z), \frac{d(z, Tz).d(Tz^*, z^*)}{d(z, z^*)}, \frac{d(z, Tz)(1+d(Tz^*, z^*))}{1+d(z, z^*)} \right\} \\ &= d(z, z^*). \end{aligned}$$

So, equation (3.10) implies

$$d(z, z^*) = d(Tz, Tz^*) \leq \beta(d(z, z^*))d(z, z^*)$$

$$\Rightarrow (d(z, z^*)) = 1$$

$$\Rightarrow (z, z^*) = 0 \Rightarrow z = z^*.$$

Corollary 3.3.(Farhan *et al.* [2]) Let (X, d) be a complete RMS and $T : X \rightarrow X$ be an α –admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(d(Tx, Ty) + l)^{\alpha(x, Tx)\alpha(y, Ty)} \leq \beta(d(x, y))d(x, y) + l$$

for all $x, y \in X$ where $l \geq 1$. Suppose that if T is continuous and there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq 1$, then f has a fixed point.

Proof: Taking $(x, y) = d(x, y)$ in Theorem 3.1, one can get the proof.

Corollary 3.4. (Farhan *et al.*[2]) Assume that all the hypotheses of Corollary 3.3 hold. Adding the following condition:

(a) If $x = Tx$, then $(x, Tx) \geq 1$,

we obtain the uniqueness of the fixed point of T .

Proof: Taking $(x, y) = d(x, y)$ in Corollary 3.3.

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