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On Sum Neighbourhood Degree Centrality of Graphs

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Abstract:

Centrality measures play a pivotal role in understanding the importance and influence of vertices within a graph. In this paper, we introduce a novel form of centrality measure called the sum neighbourhood degree centrality which is designed to quantify the structural importance of vertices in complex networks. By incorporating both its immediate connectivity and the structural characteristics of its neighbourhood, the proposed measure captures local and relational properties of a vertex more effectively than the existing centrality measures. We formally define the measure, establish its mathematical properties, and compare its behaviour with other measures using analytical results. This centrality is shown to provide improved discrimination among vertices and yields meaningful insights in to various classes of graphs and real-world networks. The main purpose of this paper is to build a theoretical property of sum neighbourhood degree centrality of vertices and sum neighbourhood degree centrality weight of graphs. Keywords: centrality measures; sum neighbourhood degree centrality; sum neighbourhood degree centrality weight.

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1. Introduction

A common set of online or offline connections between individuals or organizations or between individuals and organizations, can be considered as a network. For example, farmers' networks, business networks, and social networks like WhatsApp, Facebook, LinkedIn, and X.com. Nowadays, social networks have become part and parcel of human life. Network science provides a powerful framework for modelling complex systems in which entities are represented as vertices and their interactions as edges. Within this framework, centrality measures play a fundamental role in identifying influential or structurally important vertices. Among the various centrality metrics proposed in the literature, degree based centrality measures are among the most intuitive and widely used due to their simplicity and strong interpretability. Degree based centrality quantifies the importance of a vertex by counting the number of direct connections it has to other vertices in the network. The underlying assumption is that vertices with a higher number of immediate neighbours are more likely to exert influence, facilitate information flow, or play a critical role in local network dynamics. This notion is particularly relevant in real world networks such as social networks, biological interaction networks, communication systems, and transportation infrastructures, where local connectivity often determines functional significance. There are various centrality measures developed and applied in these networks. Here we introduced a novel form of centrality measure called the sum neighbourhood degree centrality which is designed to quantify the structural importance of vertices in complex networks. This study focuses on examining their theoretical properties, practical relevance, and effectiveness in identifying influential vertices.

Let G be a simple, connected, finite and undirected nontrivial graph with vertex set $V(G)$ and edge set $E(G)$. Further, we denote the number of vertices and edges of G by $n = |V|$ and $m = |E|$, respectively. If $u, v \in V(G)$ are adjacent then we denote it by $u \sim v$, the degree of v denoted by $d(v)$ or $d_G(v)$ is defined as the number of edges incident with v and if all vertices have the same degree k then G is called regular graph of degree k . The neighbourhood of a vertex $u \in V(G)$ is the set $N(u)$

consisting of all vertices v which are adjacent to u . The vertices and edges of G are called elements of G . The elements of G are neighbours if they are either incident or adjacent. The vertices of line graph $L(G)$ are the edges of G with two vertices of line graph adjacent whenever the corresponding edges of G are adjacent. The complement of G , denoted as \bar{G} is a graph with vertex set same as that of G and two vertices in \bar{G} are adjacent if and only if they are not adjacent in G and G is said to be self complementary graph if G is isomorphic to complement of G . A graph is acyclic if it has no cycles. A tree is a connected acyclic graph. [1, 3, 4].

Definition 1.1. [9] Degree centrality is a measure based on the number of connections a vertex has. The degree centrality of a vertex $v \in V(G)$ is defined as the number of vertices adjacent to v , which is nothing but the degree of v . This value is divided by the maximum possible degree of a vertex to normalise it. Hence, the normalised degree centrality of v is given by $DC(v) = \frac{d(v)}{n-1}$. The degree centrality weight $DCW(G)$ of G is defined as

$$DCW(G) = \sum_{k=1}^n DC(v_k)$$

Definition 1.2. [7] The closeness centrality measure is one of the three classic centrality measures at the vertex level. If the order of G is n and if $u \in V(G)$, then the closeness centrality of u is given by $C_G(u) = \frac{n-1}{T_G(u)}$

where $T_G(u) = \sum_{x \in V(G)} d_G(ux)$.

Definition 1.3. [7] Let $G = (V, E)$ be a connected graph of order n . The closeness centrality weight of G , denoted by $CW(G)$, is defined as $CW(G) = \sum_{u \in V(G)} f(u)$, where $f: V(G) \rightarrow (0, 1]$ is a function and $f(u) = C_G(u)$.

Definition 1.4. [9] Leverage centrality is a measure of the relationship between degree of a given vertex v and the degree of each of its neighbours, averaged over all neighbours, and is defined as

$$l(v) = \frac{1}{d(v)} \sum_{u \in N(v)} \frac{d(v)-d(u)}{d(v)+d(u)}$$

The leverage centrality weight $LW(G)$ of G is defined as the sum of the leverage centralities of all vertices in the graph.

Definition 1.5. [2] Let $G = (V, E)$ be a graph, and α, β be two elements of $V(G) \cup E(G)$. The generalized transformation graph G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$. α and β are adjacent in G^{ab} if and only if one of the following conditions holds:

1. $\alpha, \beta \in V(G)$, α, β are adjacent in G if $a = +$ and α, β are not adjacent in G if $a = -$.
2. $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $b = +$ and α, β are not incident in G if $b = -$.

The vertex v of G^{ab} is called point vertex (respectively, line vertex) of G^{ab} if it corresponds to a vertex (respectively, edge) of G .

Proposition 1.6. [2] Let G be an (n,m) -graph. If v and e are point vertex and line vertex in G^{ab} , then the degree of a vertex in G^{ab} is

1. $d_{G^{++}}(v) = 2d_G(v)$ and $d_{G^{++}}(e) = 2$.
2. $d_{G^{+-}}(v) = m$ and $d_{G^{+-}}(e) = n - 2$.

- 3. $d_{G^{++}}(v) = n - 1$ and $d_{G^{++}}(e) = 2$.
- 4. $d_{G^{--}}(v) = n + m - 1 - 2d_G(v)$ and $d_{G^{--}}(e) = n - 2$.

2. Main Results

Definition 2.1. Let $G = (V, E)$ be a connected graph of order n , the sum neighbourhood degree centrality of vertex u of G is defined as, $S_G(u) = \frac{S_u}{(n-1)^2}$, where $S_u = \sum_{v \in N(u)} d(v)$. And the sum neighbourhood degree centrality weight of G is defined as, $SW(G) = \sum_{u \in V(G)} S_G(u)$.

For example, Figure 1 shows the graph G , S_u , and $S_G(u)$, for every $u \in V(G)$. We can see that $SW(G)=2$.



Figure1: Graph G and corresponding values of S_u , $S_G(u)$ and $SW(G)$.

Theorem 2.2. Let G be a graph of order $n \geq 2$ and let $u \in V(G)$. Then $\frac{1}{(n-1)^2} \leq S_G(u) \leq 1$.

Proof. Let $u \in V(G)$. By definition of S_u , we have $1 \leq S_u \leq (n-1)^2$, since the maximum possible contribution arises when $d(v) = n-1$, for all $v \in N[u]$. Dividing throughout by $(n-1)^2$, we obtain

$$\frac{1}{(n-1)^2} \leq \frac{S_u}{(n-1)^2} \leq 1. \text{ Hence, } \frac{1}{(n-1)^2} \leq S_G(u) \leq 1. \text{ That is } G \sim Kn.$$

Corollary 2.3. For a graph G of order $n \geq 2$, the corresponding measure satisfies $\frac{n}{(n-1)^2} \leq SW(G) \leq n$.

Proof. Summing the inequality $\frac{1}{(n-1)^2} \leq S_G(u) \leq 1$ over all vertices $u \in V(G)$, we get

$$\sum_{u \in V(G)} \frac{1}{(n-1)^2} \leq \sum_{u \in V(G)} S_G(u) \leq \sum_{u \in V(G)} 1. \text{ Thus, } \frac{n}{(n-1)^2} \leq SW(G) \leq n.$$

Observation 2.4. 1. If $G = K_n$ with $n \geq 2$, then $SW(G) = DCW(G)$.

2. For any connected graph G , $SW(G) \leq DCW(G) \leq CW(G)$, and bounds are sharp when $G = K_n$ $n \geq 3$.

Observation 2.5. For any connected graph G of order n such that \bar{G} is also connected, then the following bounds hold:

- 1. $SW(G) + LW(G) \leq 2n$.
- 2. $SW(G) \cdot LW(G) \leq n^2$
- 3. $SW(G) + SW(G) \leq 2n$.

4. $SW(G) \cdot SW(G) \leq n^2$.

Theorem 2.6. If G is a k -regular (n,m) graph, then $SW(G) = \frac{nk^2}{(n-1)^2}$.

Proof. Since G is k -regular, each vertex is adjacent to exactly k vertices and the degree of every vertex is k . Hence, $S_u = \sum_{v \in N(u)} d(v) = k \cdot k = k^2$. Therefore,

$$SW(G) = \sum_{u \in V(G)} \frac{S_u}{(n-1)^2} = \sum_{u \in V(G)} \frac{k^2}{(n-1)^2} = \frac{nk^2}{(n-1)^2}$$

Theorem 2.7. Let $G = P_n$, $n \geq 5$, be a path graph. Then $SW(G) = \frac{4n-6}{(n-1)^2}$

Proof. Consider the path graph $G = P_n$, $n \geq 3$. For a vertex $u \in V(G)$, we have

$$S_u = \begin{cases} 2, & \text{if } u \text{ is a pendant vertex;} \\ 3, & \text{if } u \text{ is adjacent to a pendant vertex;} \\ 4, & \text{otherwise} \end{cases}$$

Hence,

$$SG(u) = \begin{cases} \frac{2}{(n-1)^2}, & \text{if } u \text{ is a pendant vertex;} \\ \frac{3}{(n-1)^2} & \text{if } u \text{ is adjacent to a pendent vertex;} \\ \frac{4}{(n-1)^2} & \text{otherwise} \end{cases}$$

Therefore,

$$SW(G) = 2 \left(\frac{2}{(n-1)^2} \right) + 2 \left(\frac{3}{(n-1)^2} \right) + (n-4) \left(\frac{4}{(n-1)^2} \right) = \frac{4n-6}{(n-1)^2}$$

Corollary 2.8. Let $G = K_{1,n}$ be a star graph. Then $SW(G) = \frac{n+1}{n}$.

Proof. In the star graph, central vertex has degree n and the remaining n vertices are pendant vertices.

Every vertex in G contributes $S_u = n$, and $S_G(u) = \frac{1}{n}$. Hence, $SW(G) = \sum_{u \in V(G)} \frac{1}{n} = \frac{n+1}{n}$.

Corollary 2.9. Let $G = C_n$, be a cycle graph. Then $SW(G) = \frac{4n}{(n-1)^2}$.

Proof. In the cycle graph $G = C_n$ every vertex has degree 2 and is adjacent to two vertices. Hence,

$$S_u = 2 + 2 = 4, \text{ for all } u \in V(G). \text{ Therefore, } SW(G) = \sum_{u \in V(G)} \frac{4}{(n-1)^2}$$

Theorem 2.10. Let $G = K_n$ be a complete graph on $n \geq 2$ vertices. Then $SW(K_n) = n$.

Proof. In a complete graph, every vertex is adjacent to all other vertices, hence for any $u \in V(G)$, $d(u) = n - 1$ and $N(u) = V(G) \setminus \{u\}$. The sum of degrees of all neighbours of u is $S_u = \sum_{v \in N(u)} d(v) = \sum_{v \in N(u)} (n - 1) = (n - 1)(n - 1) = (n - 1)^2$. By the definition of sum neighbourhood degree centrality.

$$S_G(u) = \frac{S_u}{(n-1)^2} = \frac{(n-1)^2}{(n-1)^2} = 1, \text{ for all } u \in V(G). \text{ Finally, summing over all vertices of } G \text{ gives}$$

$$SW(G) = \sum_{u \in V(G)} S_G(u) = \sum_{u \in V(G)} 1 = n.$$

Theorem 2.11. Let $G = W_{n+1}$ be a wheel graph on $n + 1$ vertices, where $n \geq 4$. Then $SW(G) = \frac{n+9}{n}$.

Proof. The graph G consists of a cycle C_n together with a central vertex v adjacent to all vertices of the cycle. Degree of the central vertex is $d(v) = n$, and degree of every other vertex u is $d(u) = 3$. For central vertex the neighbourhood of v consists of all n cycle vertices, each of degree 3. Hence,

$S_v = \sum_{u \in N(v)} d(u) = 3n$, and $S_G(v) = \frac{3n}{n^2} = \frac{3}{n}$. For remaining vertices, each cycle vertex u is adjacent to the central vertex of degree n and to two cycle vertices of degree 3. Thus, $S_u = n + 3 + 3 = n + 6$, and $S_G(u) = n + \frac{n+6}{n^2}$. Summing over all vertices of G , we obtain

$$SW(G) = S_G(v) + \sum_{u \in V(C_n)} S_G(u) = \frac{3}{n} + n \left(\frac{n+6}{n^2} \right) = \frac{3n+n^2+6n}{n^2} = \frac{n+9}{n}.$$

Theorem 2.12. Let $G = K_{m,n}$ be a complete bipartite graph with bipartition (X, Y) , where $|X| = m$, $|Y| = n$, and $m, n \geq 1$. Then $SW(K_{m,n}) = \frac{mn(m+n)}{(m+n-1)^2}$.

Proof. In the complete bipartite graph $K_{m,n}$, every vertex in X is adjacent to all vertices in Y , and vice versa. For vertices in X , each vertex $u \in X$ has degree $d(u) = n$, and all its neighbours lie in Y , each having degree m . Hence, $S_u = \sum_{v \in N(u)} d(v) = n \cdot m = mn$, and therefore, $S_G(u) = \frac{mn}{(m+n-1)^2}$. For vertices in Y , similarly, for any vertex $v \in Y$, we have $d(v) = m$, and all its neighbours lie in X , each of degree n . Thus, $S_v = \sum_{u \in N(v)} d(u) = m \cdot n = mn$, and hence, $S_G(v) = \frac{mn}{(m+n-1)^2}$. Since $|V(G)| = m + n$ and every vertex contributes the same value, we obtain

$$SW(K_{m,n}) = \sum_{w \in V(G)} S_G(w) = (m + n) \frac{mn}{(m+n-1)^2} = \frac{mn(m+n)}{(m+n-1)^2}.$$

Definition 2.13. [5] The p -sunlet graph, denoted by L_p , is the graph obtained from the cycle graph C_p by attaching exactly one pendant (leaf) vertex to each vertex of the cycle. Consequently, the p -sunlet graph has $2p$ vertices and $2p$ edges.

Theorem 2.14. Let $G = L_p$ be the p -sunlet graph with $p \geq 3$. Then $SW(G) = \frac{10p}{(2p-1)^2}$.

Proof. The graph G consists of a cycle C_p together with p pendant vertices, each attached to a distinct cycle vertex. Let u be a pendant vertex, its unique neighbour is a cycle vertex of degree 3. Hence, $S_u = 3$ then $S_G(u) = \frac{3}{(2p-1)^2}$. Let v be a cycle vertex, it is adjacent to two cycle vertices of degree 3 and one pendant vertex. Therefore, $S_v = 3+3+1 = 7$, then $S_G(v) = \frac{7}{(2p-1)^2}$. Since G has p pendant vertices and p cycle vertices, we obtain

$$SW(G) = p \left(\frac{3}{(2p-1)^2} \right) + p \left(\frac{7}{(2p-1)^2} \right) = \frac{10p}{(2p-1)^2}.$$

Theorem 2.15. Let $G = L_p$ be a p -sunlet graph. Then $SW(L(G)) = \frac{20p}{(2p-1)^2}$

Proof. By the definition of line graph, $L(G)$ contains $2p$ vertices. Each vertex $u \in V(L(G))$ has degree either 2 or 4. The neighbourhood degree sum of u is

$$S_u = \begin{cases} 8, & \text{if } d(u) = 2; \\ 12, & \text{if } d(u) = 4. \end{cases}$$

Hence, the sum neighbourhood degree centrality of u is

$$S_G(u) = \begin{cases} \frac{8}{(2p-1)^2}, & \text{if } d(u) = 2; \\ \frac{12}{(2p-1)^2} & \text{if } d(u) = 4. \end{cases}$$

Therefore,

$$SW(L(G)) = \frac{(8+12)p}{(2p-1)^2} = \frac{20p}{(2p-1)^2}.$$

Definition 2.16. [8] Let G_1 and G_2 be two disjoint graphs. The join of G_1 and G_2 is the graph $G = G_1 + G_2$ having vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$.

Theorem 2.17. For the join graph $K_2 + P_n$, where $n \geq 5$. Then $SW(K_2 + P_n) = \frac{2(n^2+10n-6)}{(n+1)^2}$.

Proof. Let K_2 be a complete graph with vertex set $\{u_1, u_2\}$ and P_n be a path graph with vertex set $\{v_1, v_2, \dots, v_n\}$. Let $G = K_2 + P_n$ be their join graph, then $V(G) = \{u_1, u_2, v_1, v_2, \dots, v_n\}$, so that $|V(G)| = n + 2$, and $|E(G)| = 3n$. In the join graph G , the vertices u_2 and u_1 , are adjacent to each other and to every vertex of P_n , while each vertex v_i , of P_n is adjacent to its neighbours in the path and to both u_1 and u_2 . If $u_i \in V(K_2)$, then $d_G(u_i) = n + 1$, and if $u_i \in V(P_n)$, then

$$d_G(u_i) = \begin{cases} 3, & \text{if } u_i \text{ is a pendent vertex of } P_n; \\ 4, & \text{if } u_i \text{ is not a pendent vertex of } P_n. \end{cases}$$

We consider the following cases.

Case 1. Let $d_G(u_i) = n + 1$, $i \in \{1,2\}$, then in G , u_i is adjacent to two vertices each of degree 3, one vertex of degree $n + 1$, and $(n - 2)$ vertices each of degree 4. So, $S_{u_i} = 2(3) + 1(n + 1) + (n - 2)4 = 6 + n + 1 + 4n - 8 = 5n - 1$. Therefore, $S_G(u_i) = \frac{5n - 1}{(n + 1)^2}$.

Case 2. Vertex v_i , $i \in \{1, n\}$ of degree 3 in G is adjacent to two vertices each of degree $n + 1$ and one vertex of degree 4. So, $S_{v_i} = 2(n + 1) + 1(4) = 2n + 2 + 4 = 2n + 6$. Therefore, $S_G(v_i) = \frac{2n - 6}{(n + 1)^2}$.

Case 3. For a vertex v_i of degree 4 in G , we consider the following subcases.

Subcase 3.1. Vertex v_i , $i \in \{2, n - 1\}$ of degree 4 in G is adjacent to one vertex of degree 3, two vertices of degree $n + 1$, and one vertex of degree 4. So, $S_{v_i} = 1(3) + 2(n + 1) + 1(4) = 3 + 2n + 2 + 4 = 2n + 9$. Therefore, $S_G(v_i) = \frac{2n - 9}{(n + 1)^2}$.

Subcase 3.2. Vertex v_i , $3 \leq i \leq n - 2$ of degree 4 in G is adjacent to two vertices of degree $n + 1$ and two vertices of degree 4. So, $S_{v_i} = 2(n + 1) + 2(4) = 2n + 2 + 8 = 2n + 10$. Therefore, $S_G(v_i) = \frac{2n - 10}{(n + 1)^2}$.

By computing the neighbourhood degree sums for each type of vertex and applying the definition of sum neighbourhood degree centrality weight, we obtain

$$SW(K_2 + P_n) = 2 \left[\frac{5n - 1}{(n + 1)^2} \right] + 2 \left[\frac{2n - 6}{(n + 1)^2} \right] + 2 \left[\frac{2n - 9}{(n + 1)^2} \right] + (n - 4) \left[\frac{2n - 10}{(n + 1)^2} \right] = \frac{2(n^2+10n-6)}{(n+1)^2}$$

Definition 2.18. [8] Let G and H be two disjoint graphs. The cartesian product of G and H , denoted by $G \times H$, is the graph with vertex set $V(G \times H) = V(G) \times V(H)$, where two vertices (g_1, h_1) and

(g_2, h_2) are adjacent in $G \times H$ if and only if either $g_1 = g_2$ and $h_1 h_2 \in E(H)$, or $h_1 = h_2$ and $g_1 g_2 \in E(G)$.

Theorem 2.19. Let $G = K_m \times K_n$, $m, n \geq 2$. Then $SW(G) = \frac{mn(m+n-2)^2}{(mn-1)^2}$

Proof. We have $|V(K_m \times K_n)| = mn$ and $|E(K_m \times K_n)| = \frac{mn(m+n-2)}{2}$. Since $K_m \times K_n$ is $(m+n-2)$ -regular, each vertex has $m+n-2$ neighbours, all of degree $m+n-2$. So, $S_u = \sum_{v \in N(u)} (m+n-2) = (m+n-2)(m+n-2) = (m+n-2)^2$. Hence, we have $SW(G) = \sum_{v \in V(G)} \frac{S_u}{(mn-1)^2} = \frac{mn(m+n-2)^2}{(mn-1)^2}$.

Theorem 2.20. Let $G = C_m$, $m \geq 3$ and $H = P_n$, $n \geq 5$ be graphs. Then $SW(G \times H) = \frac{2m(8n-7)}{(mn-1)^2}$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_m\}$ and $V(H) = \{u_1, u_2, \dots, u_n\}$ be the vertex sets of the graphs G and H , respectively. Then $|V(G \times H)| = mn$. Now we consider the following cases.

Case 1. A vertex u of degree 3 is adjacent to two vertices each of degree 3 and one vertex of degree 4. So, $S_u = 2(3) + 1(4) = 10$, and the total number of such vertices is $2m$.

Case 2. A vertex u of degree 4 is adjacent to one vertex of degree 3 and three vertices each of degree 4. So, $S_u = 1(3) + 3(4) = 3 + 12 = 15$, and the total number of such vertices is $2m$.

Case 3. A vertex u of degree 4 is adjacent to four vertices each of degree 4. So, $S_u = 4(4) = 16$, and the total number of such vertices is $mn - 4m$.

Therefore,

$$SW(G \times H) = \frac{2m(10) + 2m(15) + (mn-4m)16}{(mn-1)^2} = \frac{2m(8n-7)}{(mn-1)^2}.$$

Theorem 2.21. For a graph $K_2 \times P_n$, $SW(K_2 \times P_n) = \frac{2(9n-10)}{(2n-1)^2}$.

Proof. The number of vertices of the graph $K_2 \times P_n$ is $2n$ and the number of edges is $3n - 2$. We have the following three cases.

Case 1. A vertex u of degree 2 is adjacent to one vertex of degree 2 and one vertex of degree 3. So, $S_u = 1(2) + 1(3) = 5$, and the total number of such vertices is 4.

Case 2. A vertex u of degree 3 is adjacent to one vertex of degree 2 and two vertices each of degree 3. So, $S_u = 1(2) + 2(3) = 8$, and the total number of such vertices is 4.

Case 3. A vertex u of degree 3 is adjacent to three vertices each of degree 3. So, $S_u = 3(3) = 9$, and the total number of such vertices is $2n - 8$.

$$\text{Therefore, } SW(K_2 \times P_n) = \frac{4(5)}{(2n-1)^2} + \frac{4(8)}{(2n-1)^2} + \frac{(2n-8)9}{(2n-1)^2} = \frac{2(9n-10)}{(2n-1)^2}.$$

Theorem 2.22. For a graph $K_2 \times C_n$, $SW(K_2 \times C_n) = \frac{18n}{(2n-1)^2}$.

Proof. The number of vertices of the graph $K_2 \times C_n$ is $2n$ and the number of edges is $3n$. Let $V(K_2 \times C_n) = \{v_1, v_2, \dots, v_{2n}\}$. Here, the degree of each and every vertex u is 3, hence $S_u = 3(3) = 9$. Therefore,

$$SW(K_2 \times C_n) = \frac{2n(9)}{(2n-1)^2} = \frac{18n}{(2n-1)^2}.$$

3. Sum Neighbourhood Degree Centality Weight of G^{ab}

Theorem 3.1. Let $G = C_n$, $n \geq 4$, be a cycle graph, then

1. $SW(G^{++}) = \frac{20n}{(2n-1)^2}.$
2. $SW(G^{+-}) = \frac{2n(n^2 - 2n + 2)}{(2n-1)^2}.$
3. $SW(G^{-+}) = \frac{n^3 - 2n^2 - 5n}{(2n-1)^2}.$
4. $SW(G^{--}) = \frac{5n^3 - 24n^2 - 29n}{(2n-1)^2}.$

Proof. 1. In G^{++} , every point vertex u is adjacent to two point vertices each of degree four and to two line vertices each of degree two. So, $S_u = 2(4) + 2(2) = 12$. Also, every line vertex u is adjacent to two point vertices each of degree four. So, $S_u = 2(4) = 8$. So,

$$SW(G^{++}) = \frac{n(12) + n(8)}{(2n-1)^2} = \frac{20n}{(2n-1)^2}.$$

2. In G^{+-} , every point vertex u is adjacent to two point vertices each of degree n and to $(n-2)$ line vertices each of degree $(n-2)$. So, $S_u = 2(n) + (n-2)(n-2) = 2n + n^2 - 4n + 4 = n^2 - 2n + 4$. Also, every line vertex u is adjacent to $(n-2)$ point vertices each of degree n . So, $S_u = (n-2)n = n^2 - 2n$. So,

$$SW(G^{+-}) = \frac{n(n^2 - 2n + 4) + n(n^2 - 2n)}{(2n-1)^2} = \frac{2n(n^2 - 2n + 2)}{(2n-1)^2}.$$

3. In G^{-+} , every point vertex u is adjacent to $(n-3)$ point vertices each of degree $(n-1)$ and to two line vertices each of degree two. So, $S_u = (n-3)(n-1) + 2(2) = n^2 - 4n + 7$. Also, every line vertex u is adjacent to two point vertices each of degree $(n-1)$. So, $S_u = 2(n-1) = 2n - 2$. So,

$$SW(G^{-+}) = \frac{n(n^2 - 4n + 7) + n(2n - 2)}{(2n-1)^2} = \frac{n^3 - 2n^2 - 5n}{(2n-1)^2}.$$

4. In G^{--} , every point vertex u is adjacent to $(n-3)$ point vertices each of degree $(2n-5)$ and to $(n-2)$ line vertices each of degree $(n-2)$. So, $S_u = (n-3)(2n-5) + (n-2)(n-2) = 3n^2 - 15n + 19$. Also, every line vertex u is adjacent to $(n-2)$ point vertices each of degree $(2n-5)$. So, $S_u = (n-2)(2n-5) = 2n^2 - 9n + 10$. So,

$$SW(G^{--}) = \frac{n(3n^2 - 15n + 19) + n(2n^2 - 9n + 10)}{(2n-1)^2} = \frac{5n^3 - 24n^2 - 29n}{(2n-1)^2}.$$

Theorem 3.2. Let $G = P_n$, $n \geq 4$ be a path graph. Then

1. $SW(G^{++}) = \frac{5n-7}{(n-1)^2}.$
2. $SW(G^{+-}) = \frac{2n^3 - 7n^2 - 9n - 4}{(2n-2)^2}.$
3. $SW(G^{-+}) = \frac{n^3 - 2n^2 - 5n - 4}{(2n-2)^2}.$
4. $SW(G^{--}) = \frac{5n^3 - 29n^2 - 60n - 44}{(2n-2)^2}.$

Proof. 1. Contribution to degrees of vertices in G^{++} is as follows:

- The point vertex u in G^{++} corresponding to pendant vertex of P_n is adjacent to one point vertex of degree four and one line vertex of degree two. So, $S_u = 1(4) + 1(2) = 6$, and there are two such vertices in G^{++} .

- The point vertex u in G^{++} corresponding to a vertex adjacent to pendant vertex of P_n is adjacent to one point vertex of degree 2, one point vertex of degree four and two line vertices each of degree 2. So, $S_u = 1(2) + 1(4) + 2(2) = 10$, and there are two such vertices in G^{++} .

- Each of the remaining $(n - 4)$ point vertices u in G^{++} are adjacent to two point vertices each of degree four and two line vertices each of degree 2. So, $S_u = 2(4) + 2(2) = 12$.

- The line vertex u in G^{++} corresponding to pendant edge of P_n is adjacent to one point vertex of degree two and one point vertex of degree four. So, $S_u = 1(2) + 1(4) = 6$, and there are two such vertices in G^{++} .

- Each of the remaining $(n - 3)$ line vertices u in G^{++} are adjacent to two point vertices each of degree four. So, $S_u = 2(4) = 8$.

Therefore,

$$SW(G^{++}) = 2\left(\frac{6}{(2n-2)^2}\right) + 2\left(\frac{10}{(2n-2)^2}\right) + (n-4)\left(\frac{12}{(2n-2)^2}\right) + 2\left(\frac{6}{(2n-2)^2}\right) + (n-3)\left(\frac{8}{(2n-2)^2}\right) = \frac{5n-7}{(n-1)^2}.$$

2. Contribution to degrees of vertices in G^{+-} is as follows:

- The point vertex u in G^{+-} corresponding to pendant vertex of P_n is adjacent to one point vertex of degree $(n-1)$ and $(n-2)$ line vertices each of degree $(n-2)$. So, $S_u = 1(n-1) + (n-2)(n-2) = n^2 - 3n + 3$, and there are two such vertices in G^{+-} .

- The point vertex u in G^{+-} corresponding to a vertex adjacent to pendant vertex of P_n is adjacent to two point vertices each of degree $n - 1$ and $(n - 3)$ line vertices each of degree $(n - 2)$. So, $S_u = 2(n - 1) + (n - 3)(n - 2) = n^2 - 3n + 4$, and there are two such vertices in G^{+-} .

- Each of the remaining $(n - 4)$ point vertices u in G^{+-} are adjacent to two point vertices each of degree $(n - 1)$ and $(n - 3)$ line vertices each of degree $(n - 2)$. So, $S_u = 2(n - 1) + (n - 3)(n - 2) = n^2 - 3n + 4$

- Each line vertex u in G^{+-} is adjacent to $(n - 2)$ point vertices each of degree $(n - 1)$. So, $S_u = (n - 2)(n - 1) = n^2 - 3n + 2$.

Therefore,

$$SW(G^{+-}) = 2\left(\frac{n^2-3n+3}{(2n-2)^2}\right) + 2\left(\frac{n^2-3n+4}{(2n-2)^2}\right) + (n-4)\left(\frac{n^2-3n+4}{(2n-2)^2}\right) + (n-1)\left(\frac{n^2-3n+2}{(2n-2)^2}\right) = \frac{2n^3-7n^2-9n-4}{(2n-2)^2}.$$

3. Contribution to degrees of vertices in G^{-+} is as follows:

· The point vertex u in G^{-+} corresponding to pendant vertex of P_n is adjacent to $(n-2)$ point vertices each of degree $(n-1)$ and one line vertex of degree two. So, $S_u = (n-2)(n-1) + 1(2) = n^2 - 3n + 4$, and there are two such vertices in G^{-+} .

· Each of the remaining $(n-2)$ point vertices u in G^{-+} are adjacent to $(n-3)$ point vertices each of degree $(n-1)$ and two line vertices each of degree 2. So, $S_u = (n-3)(n-1) + 2(2) = n^2 - 4n + 7$.

· Any line vertex u in G^{-+} is adjacent to two point vertices each of degree $(n-1)$. So, $S_u = 2(n-1) = 2n - 2$.

Therefore,

$$\begin{aligned} SW(G^{-+}) &= 2 \left(\frac{n^2 - 3n + 4}{(2n-2)^2} \right) + (n-2) \left(\frac{n^2 - 4n + 7}{(2n-2)^2} \right) + (n-1) \left(\frac{2n-2}{(2n-2)^2} \right) \\ &= \frac{n^3 - 2n^2 - 5n - 4}{(2n-2)^2}. \end{aligned}$$

4. Contribution to degrees of vertices in G^{--} is as follows:

· The point vertex u in G^{--} corresponding to pendant vertex of P_n is adjacent to one point vertex of degree $(2n-4)$, $(n-3)$ point vertices each of degree $(2n-6)$ and $(n-2)$ line vertices each of degree $(n-2)$. So, $S_u = 1(2n-4) + (n-3)(2n-6) + (n-2)(n-2) = 3n^2 - 14n + 18$, and there are two such vertices in G^{--} .

· The point vertex u in G^{--} corresponding to a vertex adjacent to pendant vertex of P_n is adjacent to $(n-4)$ point vertices each of degree $(2n-6)$, one point vertex of degree $(2n-4)$ and $(n-3)$ line vertices each of degree $(n-2)$. So, $S_u = (n-4)(2n-6) + 1(2n-4) + (n-3)(n-2) = 3n^2 - 17n + 26$, and there are two such vertices in G^{--} .

· Each of the remaining $(n-4)$ point vertices u in G^{--} are adjacent to $(n-5)$ point vertices each of degree $(2n-6)$, two point vertices each of degree $(2n-4)$ and $(n-3)$ line vertices each of degree $(n-2)$. So, $S_u = (n-5)(2n-6) + 2(2n-4) + (n-3)(n-2) = 3n^2 - 17n + 28$.

· The line vertex u in G^{--} corresponding to pendant edge of P_n is adjacent to one point vertex of degree $(2n-4)$ and $(n-3)$ point vertices each of degree $(2n-6)$. So, $S_u = 1(2n-4) + (n-3)(2n-6) = 2(n^2 - 5n + 7)$, and there are two such vertices in G^{--} .

· Each of the remaining $(n-3)$ line vertices u in G^{--} are adjacent to two point vertices each of degree $(2n-4)$ and $(n-4)$ point vertices each of degree $(2n-6)$. So, $S_u = 2(2n-4) + (n-4)(2n-6) = 2(n^2 - 5n + 8)$.

Therefore,

$$\begin{aligned} SW(G^{--}) &= 2 \left(\frac{3n^2 - 14n + 18}{(2n-2)^2} \right) + 2 \left(\frac{3n^2 - 17n + 26}{(2n-2)^2} \right) + (n-4) \left(\frac{3n^2 - 17n + 28}{(2n-2)^2} \right) + 2 \left(\frac{2(n^2 - 5n + 7)}{(2n-2)^2} \right) \\ &\quad + (n-3) \left(\frac{2(n^2 - 5n + 8)}{(2n-2)^2} \right) = \frac{5n^3 - 29n^2 - 60n - 44}{(2n-2)^2}. \end{aligned}$$

Theorem 3.3. Let $G = K_n$, $n \geq 3$, be a complete graph, then

- $SW(G^{++}) = \frac{8n(2n-1)}{(n-1)(n+1)^2}$.

2. $SW(G^{+-}) = \frac{n^4+n^3-8n^2+8n}{(n-1)(n+2)^2}$.
3. $SW(G^{-+}) = \frac{4n(n+1)}{(n-1)(n+2)^2}$.
4. $SW(G^{--}) = \frac{n^5-4n^4+3n^3+4n^2-4n}{n^2+n-2)^2}$.

Proof. 1. In G^{++} , every point vertex u is adjacent to $(n - 1)$ point vertices each of degree $(2n - 2)$ and $(n - 1)$ line vertices each of degree 2. So, $S_u = (n - 1)(2n - 2) + (n - 1)2 = 2n(n - 1)$. Also, every line vertex u is adjacent to two point vertices each of degree $(2n - 2)$. So, $S_u = 2(2n - 2) = 4(n - 1)$.

Therefore,

$$SW(G^{++}) = \frac{n(2n^2-2n) + \frac{n(n-1)}{2}(4n-4)}{\left(\frac{n^2+n-2}{2}\right)^2} = \frac{8n(2n-1)}{(n-1)(n+1)^2}$$

2. In G^{+-} , every point vertex u is adjacent to $(n - 1)$ point vertices each of degree $\frac{n(n-1)}{2}$ and $\left(\frac{n^2-3n+2}{2}\right)$ line vertices each of degree $(n - 2)$. So, $S_u = (n - 1)\frac{n(n-1)}{2} + \left(\frac{n^2-3n+2}{2}\right)(n - 2) = \frac{2n^3-7n^2+9n-4}{2}$. Also, every line vertex u is adjacent to $(n - 2)$ point vertices each of degree $\frac{n(n-1)}{2}$. So, $S_u = (n - 2)\frac{n(n-1)}{2} = \frac{n^3-3n^2+2n}{2}$.

Therefore,

$$SW(G^{+-}) = \frac{n\left(\frac{2n^3-7n^2+9n-4}{2}\right) + \frac{n(n-1)}{2}\left(\frac{n^3-3n^2+2n}{2}\right)}{\left(\frac{n^2+n-2}{2}\right)^2} = \frac{n^4+n^3-8n^2+8n}{(n-1)(n+2)^2}$$

3. In G^{-+} , every point vertex u is adjacent to $(n - 1)$ line vertices each of degree 2. So, $S_u = (n - 1)2 = 2n - 2$. Also, every line vertex u is adjacent to 2 point vertices each of degree $(n - 1)$. So, $S_u = 2(n - 1) = 2n - 2$.

Therefore,

$$SW(G^{-+}) = \frac{n(2n-2) + \frac{n(n-1)}{2}(2n-2)}{\left(\frac{n^2+n-2}{2}\right)^2} = \frac{4n(n+1)}{(n-1)(n+2)^2}$$

4. In G^{--} , every point vertex u is adjacent to $\left(\frac{n^2-3n+2}{2}\right)$ line vertices each of degree $(n - 2)$. So, $S_u = \left(\frac{n^2-3n+2}{2}\right)(n - 2) = \frac{n^3-5n^2+8n-4}{2}$. Also, every line vertex u is adjacent to $(n - 2)$ point vertices each of degree $\left(\frac{n^2-3n+2}{2}\right)$. So, $S_u = (n - 2)\left(\frac{n^2-3n+2}{2}\right) = \frac{n^3-5n^2+8n-4}{2}$.

Therefore,

$$SW(G^{--}) = \frac{\left(\frac{n^2+n}{2}\right)\left(\frac{n^3-5n^2+8n-4}{2}\right)}{\left(\frac{n^2+n-2}{2}\right)^2} = \frac{n^5-4n^4+3n^3+4n^2-4n}{(n^2+n-2)^2}$$

4. Conclusion

In this paper, we compute the precise value of sum neighbourhood degree centrality of vertices in different families of graphs and sum neighbourhood degree centrality weight of graph operations and generalised transformation graph Gab of some standard graph.

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