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A Generalized Fixed Point Result on G-Contraction of CCMS

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Abstract: *In this paper, we obtain a generalized results of G- contraction in CCMS (Complete Cone Metric Spaces) . These results are generalization and extension results of some of the well known results existing in the literature.*

Keywords: *Cone metric space , contractive condition, fixed point, normal cone, R-contraction.*

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1. INTRODUCTION

In the fixed point theory the Banach contraction principle was the first and foremost theorem. In recent times Huang and Zhang [3] was generalized the metric space into CM(Cone Metric) space where the set of real numbers was replaced by an ordered Banach space and obtained some fixed point results in CMS(Cone Metric Space). Later on so many authors have been studying, and generalizing these results, and they have been extending the results in different ways and types of contractive conditions (see for e.g. [1-2, 5-8]) . Beiranvand, Moradi., et.al.[2] introduced the class of T-contraction and T-contractive functions, extending the Banach contraction principle and Edelstein's fixed point theorems. S.Moradi [6] introduced the T-Kannan contractive mapping. Recently, J.Morals and E.Rojas obtained some fixed point theorems of T-contractive mappings. In this paper we obtained a result of G-contraction on CCMS (Complete Cone Metric Spaces). Our results are generalizing and extending the results of [5].

2. PRELIMINARIES

The following are needed to obtain our main result which are due to [3, 5].

Definition .1.1. Let E be always a real Banach space and Q is a subset of E, Q is called a cone if and only if :

(i) Q is closed , non empty and $Q \neq \{0\}$,

(ii) $\alpha x + \beta y \in Q$ for all $x, y \in Q$ and $\alpha, \beta \in R$,

(iii) $x \in Q$ and $-x \in Q$ implies $x = 0$.

Definition 1.2. Given a cone $Q \subset E$, we define a partial ordering $\leq$ on E with respect to Q by $x << y$ if $x < y$ and $x \neq y$. we shall write $x << y$ if $y-x \in$ interior of Q . The cone Q is called normal if there is a number $K > 0$ such that for all $x,y \in L$,

$$0 \leq x \leq y \text{ implies } \| x \| \leq L \| y \|.$$

The least positive number L satisfying the above is called the normal constant of Q. In the following we always suppose E is a Banach space, Q is a cone in E with int $Q \neq \emptyset$ and $\leq$ is partial ordering with respect to Q.

Definition 1.3. Let ' X' be a non- empty set . And suppose that the mapping $\rho : X \times X \to E$ satisfies the following:

(a) $\rho(x, y) > 0$ for all $x, y \in X$, $x \neq y$, and $\rho(x, y) = 0$ if and only if $x = y$;

(b) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$,

(c) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

For all $x, y, , z \in X$ .Then ρ is called a cone rectangular metric on X and (X, ρ) is called a CMS(Cone Metric Space).

Definition 1.4. Let (X, ρ) be a CMS(Cone Metric Space). Let $\{x_n\}$ be a sequence in X and x ϵ X. If for every c ϵE with 0<<M there is an n_0ϵN such that $\rho(x_n , x) << c$, then $\{x_n\}$ is said to be convergence , $\{x_n\}$ convergences to x and x is the limit point of $\{x_n\}$. We denote this by $x_n \to x$, as $n \to \infty$.

Definition 1.5 .Let (X, ρ) be a CMS(Cone Metric Space). Let $\{x_n\}$ be a sequence in X and x ϵ X. If for every c ϵM with 0<< E there is an m,nϵN such that that $\rho(x_n , x_m) << c$, then $\{x_n\}$ is said to be a Cauchy sequence.

Definition 1.6 Let (X, ρ) be a CMS(Cone Metric Space). If every Cauchy sequence is convergent in X, then X is called a CCMS (Complete Cone Metric Space.

We introduce the following definition

Definition. Let (X, ρ) be a CMS and A,B : X→X two functions . A mapping B is said to be a G-contraction

$\rho(ABx , ABy) \leq a_1 \rho(Ax, Ay)+ a_2 \rho(Ax , ABx) + a_3 \rho(Ay, ABy) + a_4 \rho(Ax, ABy) + a_5 \rho(Ay, ABx)$,

for all x,yϵX and where, $a_1, a_2 , a_3 , a_4 , a_5 \geq 0$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$.

3. MAIN RESULTS

Theorem 2.1. Let (X, ρ) be a CCMS(Complete Cone Metric Space). Let " Q " be a normal cone with normal constant " L". Suppose that a mapping A; X→X is one –to –one continuous function and B; X→X a G-contraction. Then

(i).For every x_0ϵX , $\lim_{n\to\infty} \rho(AB^n x_0, AB^{n+1} x_0) = 0$.

(ii).There is qϵX such that $\lim_{n\to\infty} AB^n x_0 = 0$.

(iii). If A is subsequently convergent then $\{B^n x_0\}$ has a convergent subsequence.

(iv) . There is a unique pϵX such that Bp = p.

(v). If A is sequentially convergent then for each x_0ϵX the iterative sequence $(B^n x_0)$ converges to p.

Proof: Let x_0 be an arbitrary point in X .We define the iterative sequence $\{x_n\}$ by $x_{n+1} = Bx_n = B^n x_0$. We have

$\rho(Ax_n , A x_{n+1}) = \rho(ABx_{n-1} , ABx_n)$

$\leq a_1 \rho(Ax_{n-1} , Ax_n)+ a_2 \rho(Ax_{n-1} , A Bx_{n-1}) + a_3 \rho(Ax_n, ABx_n) + a_4 \rho(Ax_{n-1}, ABx_n) + a_5 \rho(Ax_n, ABx_{n-1})$,

$\leq a_1 \rho(Ax_{n-1} , Ax_n)+ a_2 \rho(Ax_{n-1} , A x_n) + a_3 \rho(Ax_n, Ax_{n+1}) + a_4 \rho(Ax_{n-1}, Ax_{n+1}) + a_5 \rho(Ax_n, Ax_n)$,

$\leq (a_1 + a_2 + a_4) \rho(Ax_{n-1}, Ax_n) + (a_3 + a_4)\rho(Ax_n, Ax_{n+1})$,

$\rho(Ax_n , A x_{n+1}) \leq (a_1 + a_2 + a_4) / 1- (a_3 + a_4) \rho(Ax_{n-1}, Ax_n)$,

$\leq h \rho(Ax_{n-1}, Ax_n)$, where $h = (a_1 + a_2 + a_4) / 1- (a_3 + a_4) < 1$.

And we conclude that by repeating the above argument, we get that

$\rho(AB^n x_0 , AB^{n+1} x_0) \leq h^n \rho(Ax_0 , AB x_0)$. … (1)

From (1) we have

$\| \rho(AB^n x_0 , AB^{n+1} x_0) \| \leq h^n L \| \rho(Ax_0 , AB x_0) \|$,

where L is the normal constant of M. By inequality above we get that

$\lim_{n\to\infty} \rho(AB^n x_0, AB^{n+1} x_0) = 0$. … (2)

By (1) for every m,n ϵN with m >n we have

$\rho(Ax_n , Ax_m) \leq \rho(Ax_n , Ax_{n+1})+ \ldots + \rho(Ax_{m-1} , Ax_m)$

$\leq [h^n + \ldots + h^{m-1}] \rho(Ax_0 , ABx_0)$

$\leq h^n /1-h \rho(Ax_0 , ABx_0)$.

So $\rho(AB^n x_0 , AB^{n+1} x_0) \leq h^n /1-h \rho(Ax_0 , ABx_0)$. … (3)

From (3) we have

$\| \rho(AB^n x_0 , AB^{n+1} x_0) \| \leq h^n /1-h\, L \| \rho(Ax_0 , ABx_0) \|$.

Where L is the normal constant of M .Taking limit and w have h<1, we obtain

$\lim_{n,m \to \infty} \rho(AB^n x_0, AB^m x_0) = 0$. In this way we have

$\lim_{n \to \infty} \rho(AB^n x_0, AB^m x_0) = 0$.

That implies $(AB^n x_0)$ is a Cauchy sequence in X. Since X is CCM-Space, then there exists qϵX such that $\lim_{n \to \infty} AB^n x_0 = q$. …

(4)

Now if A is sub sequentially convergent $(B^n x_0)$ has a convergent subsequence. So there is a pϵX and $\{ x_{n_r} \}$ such that $\lim_{r \to \infty} B^{n_r} x_0 = p$. …

(5)

Since A is continuous and (5) we obtain

$\lim_{r \to \infty} AB^{n_r} x_0 = Ap$. … (6)

From (4) and (6) we conclude that Ap = q. … (7)

On the other hand

$\rho(ABp , Ap) \leq \rho(ABp , A B^{n_r} x_0) + \rho(A B^{n_r} x_0 , A B^{n_r+1} x_0) + \rho(A B^{n_r} x_0 , Ap)$

$\leq a_1 \rho(Ap , AB^{n_r-1} x_0) + a_2 \rho(Ap, ABp) + a_3 \rho(A B^{n_r-1} x_0 , A B^{n_r} x_0) +$
$a_4 \rho(Ap, A B^{n_r} x_0) + a_5 \rho(A B^{n_r-1} x_0 \; ABp) +$
$h^{n_r} (Ax_0 , AB x_0) + \rho(A B^{n_r+1} x_0 \; Ap)$

$\leq a_1 \rho(Ap , AB^{n_r-1} x_0) + a_2 \rho(Ap, ABp) + a_3 \rho(A B^{n_r-1} x_0 , A B^{n_r} x_0) +$
$a_4 \rho(Ap, A B^{n_r} x_0) + a_5 \rho(A B^{n_r-1} x_0 \; Ap) + a_5 \rho(A p, \; ABp)$
$+ h^{n_r} (Ax_0 , AB x_0) + \rho(A B^{n_r+1} x_0 \; Ap)$

$1-(a_2 + a_5) \rho(ABp , Ap) \leq a_1 \rho(Ap , AB^{n_r-1} x_0) + a_3 \; h^{n_r-1} \rho(Ax_0 , A Bx_0) +$
$a_4 \rho(Ap, A B^{n_r} x_0) + a_5 \rho(A B^{n_r-1} x_0 \; Ap) +$
$h^{n_r} \; (Ax_0 , AB x_0) + \; \rho(A B^{n_r+1} x_0 \; Ap)$

$\rho(ABp , Ap) \leq a_1 \rho(Ap , AB^{n_r-1} x_0) + a_3 /1-(a_2 + a_5) \; h^{n_r-1} \rho(Ax_0 , A Bx_0) +$
$a_4 / 1-(a_2 + a_5) \rho(Ap, A B^{n_r} x_0) + a_5 / 1-(a_2 + a_5) \rho(A B^{n_r-1} x_0 \; Ap) +$
$h^{n_r} \; /1-(a_2 + a_5) \; (Ax_0 , AB x_0) +$
$1/1-(a_2 + a_5) \quad \rho(A B^{n_r+1} x_0 \; Ap)$.

Thus,

$\| \rho(ABp , Ap) \| \leq a_1 / 1-(a_2 + a_5) \, L \| \rho(Ap , AB^{n_r-1} x_0) \| +$
$a_3 /1-(a_2 + a_5) \; h^{n_r-1} L \| \rho(Ax_0 , A Bx_0) \| +$
$a_4 / 1-(a_2 + a_5) \, L \| \rho(Ap, A B^{n_r} x_0) \| + a_5 / 1-(a_2 + a_5) \, L \| \rho(A B^{n_r-1} x_0 \; Ap) \| +$
$h^{n_r} \; /1-(a_2 + a_5) L \| (Ax_0 , AB x_0) \| +$
$1/1-(a_2 + a_5) L \; \| \rho(A B^{n_r+1} x_0 \; Ap) \| \to 0$ as $n \to \infty$,

where L is the normal constant of X. The convergence above gives us that

$\rho(ABp , Ap) = 0$. Implies that ABp = Ap. Since A is one –to –one , then Bp= p , consequently B has a fixed point. Because B is a G-contraction we have

$\rho(ABp , ABq) \leq a_1 \rho(Ap , Aq) + a_2 \rho(Ap, ABp) + a_3 \rho(Aq , ABq) +$
$a_4 \rho(Ap, ABq) + a_5 \rho(A q, ABp)$

If q is another fixed point of B then from injective of B we get that Bp =Bq or which is the same the fixed point is unique. Finally A is sequentially convergent by replacing n by n_r we conclude that $\lim_{n \to \infty} B^n x_0 = p$. This shows that $(B^n x_0)$ converges to the fixed point of B.

This completes the proof of the theorem.

Remark: If we take $a_1 = a_4 = a_5 = 0$ and $a_2 = a_3 = b$ in the above theorem then we get the Theorem 3.1. of [5]

Conclusion: Our results are more generalizations results than the results of [5].

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